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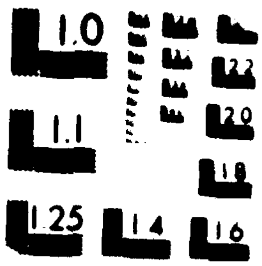
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FIRST INTERIM REPORT

Stochastic Control with Partial Observation and Approximation Techniques

I) RESEARCH

The contractors have concentrated their efforts on the design of approximation techniques in nonlinear filtering.

1 - Nonlinear filtering in the case of a high signal-to-noise ratio

The problem is roughly as follows. $\{X_t\}$ being an unobserved diffusion process, we suppose that we observe the process $\{Y_t\}$ given by

$$Y_t = \int_0^t h(X_s)\,ds + \varepsilon W_t$$

where h is one to one, $\{W_t\}$ is the "observation noise", and ε is a "small parameter". After the initial work of Bobrovsky-Katzur-Schuss, J. Picard has rigorously proved in [9] that one can design approximate filters whose difference with the optimal filter is of arbitrary order in ε. Recently, J. Picard [10] has improved his result in the sense that he does not assume anymore that the initial law of X_0 has a density. The new mathematical tool which made this improvement possible is the stochastic calculus of variations, which is a branch of the so-called "Malliavin calculus". On the other hand, A. Bensoussan [2] has given a purely analytic proof of the first version of Picard's results, thus avoiding several delicate technical tools from the theory of stochastic processes.

Two news projects have been initiated on this subject, and will be reported on with more details in the next reports:

a - E. Pardoux studies, in collaboration with W. Fleming (Brown University, USA) the case where h is only locally one to one.

b - Paula Milheiro (student of E. Pardoux) is making some numerical tests on the "Picard filters".

2 - Piecewise linear filtering

Consider a nonlinear filtering problem

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t$$

$$dY_t = h(X_t)\,dt + dV_t$$

where $\{W_t\}$ and $\{V_t\}$ are standard Wiener processes, $\{X_t\}$ and $\{Y_t\}$ are supposed for simplicity to be one dimensional, $\{X_t\}$ being unobserved and $\{Y_t\}$ observed.

We assume that we can partition $\mathbf{R}$ into a finite union of disjoint intervals $(I_1, \ldots, I_n)$ in such a way that on each of the I_i's b and h are linear, and σ is constant. We can naturally associate to the above nonlinear filtering problem n linear filtering problems. Suppose we start the Kalman filter corresponding to the i-th linear filtering problem with the nongaussian initial law which is the restriction to I_i of the law of X_0. During a "small" interval of time h, most of the "mass" stays in I_i, so that we make a small error by running the linear filter. The n-1 other Kalman filters are working similarly in parallel. At time h, the output of the n linear filters are summed up, and the sum is split according to the partition $\{I_1, \ldots, I_n\}$, which gives the initial laws for the n Kalman filters which run in parallel on the interval $[h, 2h]$, etc...

C. Savona (student of E. Pardoux) has proved in her thesis [11] (see also [8]) that the output of this procedure convergences to the optimal filter as $h \longrightarrow 0$. More recently, she has tested numerically this procedure. The first results are deceiving on some of the examples, in the sense that it seems necessary to choose h very small, for the result to be reasonably good. This point will be checked again in the near future.

3 - Numerical solution of Zakai's equation.

F. Le Gland [6] has studied in great detail the problem of the time-discretization of Zakai's equation. He suggests in particular a new scheme, whose error is of the order of $(\Delta t)^{3/2}$ if Δt is the time-discretization step. An original probabilistic interpretation of the latter scheme is provided.

4 - Parameter estimation for partially observed stochastic processes.

Fabien Campillo and François Le Gland [5] have compared the EM algorithm (proposed in the context of partially observed stochastic processes by Dembo and Zeitouni) with the standard maximum likelihood approach, which consists in maximizing the integral over the whole space of Zakai's equation. The EM algorithm seems at first sight to be more efficient, but requires a great deal of memory, since it uses a smoothing algorithm (vs. filtering). Also the number of iterations required has to be checked in practice. A numerical comparison will be done in the near future.

5 - Dynamic observers.

Nonlinear filtering, which is a stochastic theory, has a deterministic counterpart, which is the theory of "dynamic" observers. The object of this theory is to give a way of reconstructing the solution of a given differential equation with unknown initial condition, from partial observations. There are obvious connections between the theory of filtering, and the theory of observers. One of the issues of the latter is the question of observability, which is also an important practical issue in filtering.

A. Bensoussan, J. Baras and M. James [4] have shown that a dynamic observer can be viewed as the limit of stochastic filters, when the intensity of the noises tends to zero. A. Bensoussan and J. Baras [3] have also studied observers for systems governed by PDEs.

II) - TRANSFER FROM FRANCE TO THE U.S.

A. Bensoussan has given a series of "distinguished lectures" at the Systems Research Center of the University of Maryland in November of 1986, on nonlinear filtering and stochastic control with partial observation.

E. Pardoux has given in March 1987 a series of lectures in the same framework, on the applications of the Malliavin Calculus, in particular to nonlinear filtering. The Malliavin Calculus is a new branch of stochastic analysis, which has been developped essentially in France, the U.S. and Japan, by theoretical probabilists. This new tool has proved to have important applications in filtering, and its popularization among applied probabilists is now an important task.

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APPENDIX 1

Un Algorithme de Résolution Approchée Pour le Filtrage Non Linéaire par Morceaux

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UN ALGORITHME DE RESOLUTION APPROCHEE POUR LE FILTRAGE LINEAIRE PAR MORCEAUX

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Introduction

On propose une méthode de résolution approchée pour la classe des problèmes de filtrage "linéaires par morceaux": le signal $\{X_t\}_{t\geq 0}$ est solution d'une équation différentielle stochastique dont les coefficients de dérive et de diffusion sont respectivement linéaire par morceaux et constant par morceaux non dégénéré; on observe un processus $\{Y_t\}_{t\geq 0}$ de la forme

$$Y_t = \int_0^t h(X_s)ds + B_t, \qquad t \geq 0$$

avec h continue linéaire par morceaux, $\{B_t\}_{t\geq 0}$ mouvement brownien indépendant du signal et on cherche à caractériser la loi conditionnelle du signal X_T sachant la tribu des observations jusqu'à l'instant T pour tout $T > 0$ (le "filtre").

Pour un problème de filtrage linéaire avec condition initiale gaussienne, la loi conditionnelle du signal sachant les observations est une gaussienne dont la moyenne et la variance sont solutions respectivement d'une équation différentielle stochastique et d'une équation différentielle ordinaire de type Riccati, construites sur l'observation (filtre de Kalman-Bucy). En revanche on ne sait en général pas, pour un problème de filtrage non linéaire quelconque, mettre en évidence un ensemble fini de statistiques suffisantes, solutions d'un système récursif de dimension finie construit sur l'observation, permettant de calculer le filtre (si c'est le cas, on dit que le problème de filtrage est de dimension finie); la résolution directe d'un problème de filtrage non linéaire conduit donc sauf cas particuliers à un algorithme de dimension infinie. Beneš-Karatzas étudient par exemple dans [2] le problème de filtrage linéaire par morceaux dans le cas d'un signal de dimension 1 dont les coefficients de dérive et de diffusion sont de plus respectivement continu et constant. Ils obtiennent, en utilisant des techniques classiques de construction de la solution fondamentale d'une équation aux dérivées partielles de type parabolique, une représentation de la densité conditionnelle à partir d'un nombre fini de statistiques suffisantes: une partie de ces statistiques est solution d'un système récursif de dimension finie mais l'autre est solution d'un système d'équations intégrales (ces deux parties correspondent respectivement aux intervalles de linéarité et à la prise en compte des points anguleux des coefficients).

Dans [7], on a mis en évidence une suite de filtres sous-optimaux pour le problème de filtrage linéaire par morceaux, obtenus en discrétisant le temps et en exploitant le caractère linéaire par morceaux des coefficients, et on a établi la convergence de ces filtres vers le filtre optimal. On va exploiter ici ce résultat de convergence et montrer comment le problème de filtrage linéaire par morceaux peut être résolu de façon approchée par le calcul d'une batterie de filtres linéaires avec condition initiale non gaussienne; ceux-ci sont calculés à l'aide de l'algorithme proposé par Makowski dans [6].

Enfin, signalons que Di Masi-Runggaldier étudient en [3], [4] le problème de filtrage linéaire par morceaux en temps discret. Dans [4] ils proposent un filtre de dimension finie qui l'"approche" en ce sens que les moments conditionnels pour le problème linéaire par morceaux et pour ce filtre de dimension finie convergent vers la même limite lorsque les variances de la loi initiale et du bruit du signal tendent vers 0. Dans [3], ils traitent le cas particulier où les coefficients du signal sont constants par morceaux; la loi conditionnelle est alors combinaison linéaire de K gaussiennes: la moyenne, la variance de ces gaussiennes et leur nombre K sont fonction des coefficients du signal, les coefficients de la combinaison linéaire se calculent de façon récursive.

Dans §1 on donne la formulation du problème de filtrage linéaire par morceaux, des problèmes de filtrage approchés et on rappelle le résultat de convergence établi en [7] puis on étudie en §2 un algorithme de résolution approchée pour ce problème. Les courbes représentant la densité conditionnelle pour un exemple numérique sont données en annexe.

Notations. On note $C_b(\mathbb{R}^d)$ l'ensemble des fonctions continues bornées sur $\mathbb{R}^d$, on fixe un temps terminal T et on note C^d l'ensemble des fonctions continues définies sur $[0,T]$ à valeurs dans $\mathbb{R}^d$, C_0^d l'ensemble des éléments de C^d qui sont de plus nuls en 0; si X est un processus aléatoire défini sur C^d muni de sa tribu borélienne, on convient de noter $(\mathcal{F}_t^X)_{t\geq 0}$ sa filtration naturelle.

1. Formulation du problème de filtrage et de ses approximations

Soit $\{P_k,\ 1 \leq k \leq K\}$ une partition finie de $\mathbb{R}^d$ où les P_k, $k = 1,\ldots,K$ sont des polyèdres. Soit b et h deux applications de $\mathbb{R}^d$ respectivement dans $\mathbb{R}^d$, $\mathbb{R}^q$, affines sur chacun des polyèdres de la partition $\{P_k,\ 1 \leq k \leq K\}$, h étant de plus supposée continue. Soit $\sigma_1,\ldots,\sigma_K$ K matrices $d \times d$ non dégénérées et σ la fonction prenant la valeur σ_k sur P_k. Enfin, on désigne par b_k (resp. h_k) la fonction affine de $\mathbb{R}^d$ dans $\mathbb{R}^d$ (resp. dans $\mathbb{R}^q$) qui coïncide avec b (resp. h) sur P_k.

Soit $\Omega = C^d$ d'élément générique ω, $\{X_t,\ t \in [0,T]\}$ le processus canonique sur Ω et π_0 une loi de probabilité sur $\mathbb{R}^d$ absolument continue par rapport à la mesure de Lebesgue de densité p_0 admettant des moments exponentiels de tous ordres. D'après Krylov [5], le problème de martingales associé à l'équation différentielle stochastique

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \qquad t \in [0,T] \tag{1}$$

admet une solution; la fonction σ étant de plus non dégénérée et constante par morceaux sur une famille finie de polyèdres de $\mathbb{R}^d$, on a unicité en loi des solutions de (1) d'après

Bass-Pardoux [1]. On peut donc considérer $\mathbb{P}$, unique solution sur $(\Omega, \mathcal{F}^X)$ du problème de martingales associé à (1) telle que, sous $\mathbb{P}$, π_0 est la loi du vecteur aléatoire X_0. L'espérance par rapport à $\mathbb{P}$ est notée $\mathbb{E}$ et pour $s \in [0,T]$, $x \in \mathbb{R}^d$, $\mathbb{E}_{s,x}$ désigne l'espérance par rapport à la loi du processus solution de (1) avec la condition $X_s = x$.

Le problème de filtrage linéaire par morceaux $\mathcal{P}$ est le problème de filtrage avec un signal $\{X_t,\ t \in [0,T]\}$ continu à valeurs dans $\mathbb{R}^d$ de loi $\mathbb{P}$ et un processus observé

$$Y_t = \int_0^t h(X_s)ds + B_t, \qquad t \in [0,T]$$

avec $\{B_t,\ t \in [0,T]\}$ mouvement brownien indépendant du signal. Rappelons comment ce problème est construit par la méthode de la probabilité de référence. Soit (X,Y) le processus canonique sur $C^d \times C_0^q$ muni de la loi de probabilité $\mathring{\mathbb{P}} = \mathbb{P} \otimes \mathcal{W}$, $\mathcal{W}$ étant la mesure de Wiener sur C_0^q ($\mathring{\mathbb{P}}$ est la probabilité de référence). Sur $C^d \times C_0^q$ on définit le processus

$$Z_t(\omega,y) = \exp\Big\{I_t(\omega,y) - \frac{1}{2}\int_0^t \big|h(X_s(\omega))\big|^2 ds\Big\}, \qquad t \in [0,T]$$

où pour tout $\omega \in \Omega$, $I_t(\omega,\cdot)$ est $\mathcal{W}$ indistingable de l'intégrale stochastique $\int_0^t h(X_s(\omega))dY_s$. Les hypothèses faites sur b, σ et h assurent que $\{Z_t\}_t$ est une $(\mathcal{F}_t^{X,Y}, \mathring{\mathbb{P}})$ martingale. Soit maintenant $\widetilde{\mathbb{P}}$ la loi de probabilité sur $C^d \times C_0^q$ définie par

$$\left.\frac{d\widetilde{\mathbb{P}}}{d\mathring{\mathbb{P}}}\right|_{\mathcal{F}_T^{X,Y}} = Z_T.$$

Résoudre le problème $\mathcal{P}$ consiste à calculer pour tout $t \in [0,T]$ Γ_t, loi de probabilité conditionnelle régulière de X_t sachant $\mathcal{F}_t^Y$ sous $\widetilde{\mathbb{P}}$, appelée "filtre" ou "filtre normalisé" à l'instant t. Pour tout élément y de C_0^q, introduisons la fonction à valeurs mesures $\{\mu_t^{\mathbb{P},y},\ t \in [0,T]\}$ défini par

$$\forall t \in [0,T] \qquad \forall \phi \in C_b(\mathbb{R}^d) \qquad \langle \mu_t^{\mathbb{P},y}, \phi\rangle = \mathbb{E}\big[\phi(X_t)Z_t(y)\big].$$

D'après la formule de Kallianpur-Striebel, pour tout ϕ dans $C_b(\mathbb{R}^d)$, on a les égalités $\mathcal{W}$ p.s.

$$\langle \Gamma_t, \phi\rangle = \frac{\mathring{\mathbb{E}}\big[\phi(X_t)Z_t \mid \mathcal{F}_t^Y\big]}{\mathring{\mathbb{E}}\big[Z_t \mid \mathcal{F}_t^Y\big]} = \frac{\langle \mu_t^{\mathbb{P},\cdot}, \phi\rangle}{\langle \mu_t^{\mathbb{P},\cdot}, 1\rangle}.$$

Le processus $\{\mu_t^{\mathbb{P},\cdot},\ t \in [0,T]\}$ est appelé le filtre non normalisé (solution de l'équation de Zakaï) et la formule de Kallianpur-Striebel exprime que la donnée du filtre non normalisé permet de caractériser le filtre Γ_T. On introduit également le processus $\mathcal{F}_t^Y$ adapté à valeurs mesures $q(t,dz,s,x)$, $0 \le s \le t \le T$, $x \in \mathbb{R}^d$ qui, pour une valeur y de l'observation est défini par

$$\forall \phi \in C_b(\mathbb{R}^d) \qquad \int \phi(z)q(t,dz,s,x) = \mathbb{E}_{s,x}\big[\phi(X_t)Z_t(y)\big].$$

Le processus $q(t, dz, 0, x)$ est la solution fondamentale de l'équation de Zakaï pour $\mathcal{P}$ et on a

$$\forall \phi \in C_b(\mathbb{R}^d) \qquad \langle \mu_t^{\mathbb{P},\cdot}, \phi \rangle = \int \Big\{ \int \phi(z) q(t, dz, 0, x) \Big\} p_0(x) dx$$
$$= \int \phi(z) \Big\{ \int q(t, dz, 0, x) p_0(x) dx \Big\}$$

Construisons maintenant la famille $\{\mathcal{P}^n\}_n$ des problèmes de filtrage approchés. Pour cela on se donne une suite $\{\mathcal{T}_T^n\}_n$ de subdivisions de $[0,T]$ $t_0^n = 0 < t_1^n < \ldots < t_n^n = T$; on pose pour $n \in \mathbb{N}$, $s \in [0,T]$

$$|\mathcal{T}_T^n| = \max\big\{ |t_{j+1}^n - t_j^n|, \quad j = 0, 1, \ldots, n-1 \big\},$$

$$I_s^n = \big\{ j = 0, 1, \ldots, n, \quad t_j^n \le s \big\}, \qquad n(s) = \max I_s^n, \qquad \bar{s}^n = t_{n(s)}^n$$

et on suppose que $|\mathcal{T}_T^n| \to 0$ quand $n \to \infty$. Pour tout n, on désigne par b^n (resp. σ^n, h^n) la fonctionnelle sur $[0,T] \times \Omega$ définie par

$$\forall (s, \omega) \in [0,T] \times \Omega \qquad b^n(s, \omega) = \sum_{k=1}^{K} 1_{P_k}\big(\omega(\bar{s}^n)\big) b_k\big(\omega(s)\big),$$
$$h^n(s, \omega) = \sum_{k=1}^{K} 1_{P_k}\big(\omega(\bar{s}^n)\big) h_k\big(\omega(s)\big),$$
$$\sigma^n(s, \omega) = \sum_{k=1}^{K} 1_{P_k}\big(\omega(\bar{s}^n)\big) \sigma_k.$$

On a existence et unicité trajectorielle des solutions du problème de martingales associé à l'équation différentielle stochastique

$$dX_t = b^n(t, X) dt + \sigma^n(t, X) dW_t \tag{2_n}$$

et on désigne par $\mathbb{P}^n$ l'unique loi de probabilité sur $(\Omega, \mathcal{F}^X)$ solution de ce problème avec la loi initiale π_0. On note $\mathbb{E}^n$ l'espérance par rapport à $\mathbb{P}^n$ et $\mathbb{E}_{s,x}^n$, $s \in [0,T]$, $x \in \mathbb{R}^d$ l'espérance par rapport à la loi du processus solution de (2_n) avec la condition $X_s = x$. On prend pour problème de filtrage $\mathcal{P}^n$ le problème avec un signal $\{X_t, \, t \in [0,T]\}$ continu à valeurs dans $\mathbb{R}^d$ de loi $\mathbb{P}^n$ et un processus observé de la forme

$$Y_t = \int_0^t h^n(s, X) ds + B_t^n, \qquad t \in [0,T]$$

avec $\{B_t^n, \, t \in [0,T]\}$ mouvement brownien indépendant du signal. Toujours d'après la formule de Kallianpur-Striebel, le filtre normalisé Γ_T^n à l'instant T pour le problème $\mathcal{P}^n$ est caractérisé par la donnée du filtre non normalisé $\{\mu_T^{\mathbb{P}^n, y}, \, y \in C_0^q\}$ donné par

$$\forall \phi \in C_b(\mathbb{R}^d) \qquad \langle \mu_T^{\mathbb{P}^n, y}, \phi \rangle = \mathbb{E}^n \big[\phi(X_t) Z_T^n(y) \big],$$

$$Z_t^n(\omega, y) = \exp\left\{ I_t^n(\omega, y) - \frac{1}{2}\int_0^t |h^n(s,\omega)|^2 ds \right\}$$

où pour tout $\omega \in \Omega$, $I_t^n(\omega, \cdot)$ est $\mathcal{W}$ indistingable de l'intégrale stochastique $\int_0^t h^n(s, X) dY_s$. Enfin, on introduit comme ci-dessus les processus à valeurs mesures $q^n(t, dz, s, x)$, $0 \leq s \leq t \leq T$, $x \in \mathbb{R}^d$ qui pour une valeur y de l'observation sont définis par

$$\forall \phi \in C_b(\mathbb{R}^d) \qquad \int \phi(z) q^n(t, dz, s, x) = \mathbb{E}_{s,x}^n [\phi(X_t) Z_t^n(y)].$$

Pour n fixé, le problème $\mathcal{P}^n$ est tout comme le problème initial $\mathcal{P}$ un problème de filtrage non linéaire; il n'est pas non plus de dimension finie mais il possède une propriété intéressante: pour tout x dans $\mathbb{R}^d$, pour tout $j = 0, 1, \ldots, n-1$, $\mathcal{P}^n$ est conditionnellement linéaire sachant $X_{t_j^n} = x$ sur l'intervalle $[t_j^n, t_{j+1}^n[$. D'autre part, on a établi dans [7] la convergence étroite de la suite $\{\mu_T^{\mathbb{P}^n, y}\}_n$ vers $\mu_T^{\mathbb{P}, y}$ uniformément sur les parties compactes de C_0^q. Dans ce qui suit, on va utiliser cette propriété des $\mathcal{P}^n$ et le résultat de convergence pour calculer une approximation de la solution du problème $\mathcal{P}$. On note désormais $\mu_t(dz)$, $\mu_t^n(dz)$ pour $\mu_t^{\mathbb{P}, y}$, $\mu_t^{\mathbb{P}^n, y}$.

2. Présentation d'un algorithme de résolution approchée pour $\mathcal{P}$

On suppose pour simplifier les notations que $\{\mathcal{T}_T^n\}$ est la suite de subdivisions régulières de $[0, T]$ de pas $\delta_n = T/n$. Pour $k = 1, \ldots, K$, on introduit l'équation différentielle stochastique

$$dX_t = b_k(X_t)dt + \sigma_k dW_t, \qquad t \in [0, T]. \tag{3_k}$$

Pour tout x dans $\mathbb{R}^d$, s dans $[0, T]$, on désigne par $\mathbb{E}_{k,s,x}$ l'espérance par rapport à la loi du processus solution de (3_k) avec la condition $X_s = x$ et on définit pour une valeur fixée y de l'observation les processus à valeurs mesures $q_k(t, dz, s, x)$, $0 \leq s \leq t \leq T$ par

$$\forall \phi \in C_b(\mathbb{R}^d) \qquad \int \phi(z) q_k(t, dz, s, x) = \mathbb{E}_{k,s,x} [\phi(X_t) Z_k(t)(\cdot, y)]$$

avec

$$Z_k(t)(\omega, y) = \exp\left\{ I_{k,t}(\omega, y) - \frac{1}{2}\int_0^t |h_k(X_s(\omega))|^2 ds \right\}, \qquad (\omega, y) \in C^d \times C_0^q$$

où pour tout $\omega \in \Omega$, $I_{k,t}(\omega, \cdot)$ est $\mathcal{W}$ indistingable de l'intégrale stochastique $\int_0^t h_k(X_s) dY_s$. On fixe $n \in \mathbb{N}$ et on note δ pour δ_n; on a alors, pour $j = 0, 1, \ldots, n-1$

$$\begin{aligned}
\mu_{(j+1)\delta}^n(dz) &= \int q^n((j+1)\delta, dz, j\delta, x) \mu_{j\delta}^n(dx) \\
&= \sum_{k=1}^K \int_{P_k} q_k((j+1)\delta, dz, j\delta, x) \mu_{j\delta}^n(dx)
\end{aligned}$$

ce qui nous permet de calculer par récurrence une approximation $Q^{\delta}(T, dz)$ de $\mu_T(dz)$ en procédant de la façon suivante. On pose $Q(0, dz) = p_0(z)dz$, supposons calculée l'approximation $Q(j\delta, dz)$ de $\mu_{j\delta}(dz)$. Alors $\mu_{(j+1)\delta}(dz)$ s'approche par

$$Q((j+1)\delta, dz) = \sum_{k=1}^{K} \int_{P_k} q_k((j+1)\delta, dz, j\delta, x)Q(j\delta, dx).$$

La mesure $Q(j\delta, dz)$ étant supposée connue, le calcul de $Q((j+1)\delta, dz)$ se ramène au calcul des quantités

$$A_k((j+1)\delta, dz) = \int_{P_k} q_k((j+1)\delta, dz, j\delta, x)Q(j\delta, dx).$$

Or $A_k((j+1)\delta, dz)$ s'interprète comme le filtre non normalisé à l'instant $(j+1)\delta$ pour le problème de filtrage $\mathcal{P}_k$ déduit de $\mathcal{P}$ en remplaçant b, σ, h par b_k, σ_k, h_k, avec la loi initiale $1_{P_k}Q(j\delta, dz)$ à l'instant $j\delta$, c'est-à-dire comme un filtre linéaire avec condition initiale non gaussienne que l'on sait implémenter. En effet, Makowski a obtenu dans [6] le résultat suivant: la loi conditionnelle à un instant t donné pour un problème de filtrage linéaire avec un signal de loi initiale $Q_0(dz)$ non gaussienne peut se calculer en intégrant par rapport à $Q_0(dz)$ un noyau qui dépend des sorties à l'instant t d'un système récursif de dimension finie auxiliaire construit à partir des coefficients du problème de filtrage (et dans lequel la loi initiale $Q_0(dz)$ n'intervient pas). On appelle SA_k le système auxiliaire ainsi associé au problème de filtrage linéaire $\mathcal{P}_k$ pour $k = 1, \ldots, K$.

Détaillons la procédure dans le cas où le signal $\{X_t\}_t$ est un mouvement brownien réel avec une loi initiale $F(dz)$ centrée de densité p_0 et le processus observé de la forme

$$Y_t = \int_0^t X_s\, ds + B_t$$

avec $\{B_t\}_t$ mouvement brownien réel indépendant du signal. Introduisons le problème de filtrage $\mathcal{P}^+$ (resp. $\mathcal{P}^-$) avec un signal $\{X_t\}_t$ brownien réel et un processus observé de la forme

$$\int_0^t X_s ds + B_t \qquad \left(\text{resp.} \qquad \int_0^t X_s ds + B_t\right).$$

On note $A^+(\delta, dz)$ (resp. $A^-(\delta, dz)$) le filtre non normalisé à l'instant δ pour le problème $\mathcal{P}^+$ (resp. $\mathcal{P}^-$) avec la loi initiale de densité $1_{\mathbb{R}^+}(x)p_0(x)dx$ (resp. $1_{\mathbb{R}^-}(x)p_0(x)dx$). On approche $\mu_\delta(dz)$ par

$$Q(\delta, dz) = A^+(\delta, dz) + A^-(\delta, dz)$$

puis on réitère la procédure ci-dessus entre l'instant δ et l'instant 2δ avec $F(dz) = Q(\delta, dz)$ comme nouvelle loi initiale et ainsi de suite jusqu'à l'instant final T; la mesure $Q(j\delta, dz)$ étant l'approximation de $\mu_{j\delta}(dz)$ ainsi calculée, $\mu_{(j+1)\delta}(dz)$ s'approche par

$$Q((j+1)\delta, dz) = A^+((j+1)\delta, dz) + A^-((j+1)\delta, dz)$$

où $A^+((j+1)\delta, dz)$ (resp. $A^-((j+1)\delta, dz)$) est le filtre non normalisé à l'instant $(j+1)\delta$ pour le problème $\mathcal{P}^+$ (resp. $\mathcal{P}^-$) avec la loi initiale à l'instant $j\delta$

$$F_j^+(dz) = 1_{\mathbb{R}^+}(z)Q(j\delta, dz) \qquad (\text{resp. } \; F_j^-(dz) = 1_{\mathbb{R}^-}(z)Q(j\delta, dz))$$

et les résultats de Makowski [6] fournissent la valeur en tout point de $\mathbb{R}$ des densités de ces filtres. Plus précisément, soit $\{\xi_t^\pm\}_t$, $\{\varsigma_t^\pm\}_t$ les solutions des équations différentielles stochastiques

$$\begin{aligned} d\xi_t^\pm &= \pm P(t)[dY_t - (\pm\xi_t^\pm dt)], \qquad \xi_0^\pm = 0, \\ d\varsigma_t^\pm &= \pm(R(t)-1)[dY_t - (\pm\varsigma_t^\pm dt)], \qquad \varsigma_0^\pm = 0, \end{aligned}$$

$P(t)$, $R(t)$ étant solutions des équations différentielles ordinaires de type Riccati

$$\begin{aligned} \frac{dP(t)}{dt} &= -P(t)^2 + 1, \qquad P(0) = 0, \\ \frac{dR(t)}{dt} &= -P(t)(1 - R(t)), \qquad R(0) = 0. \end{aligned}$$

Alors, en définissant $S(t)$ par

$$S(t) = \int_0^t \left[1 - (R(s)-1)^2\right] ds,$$

pour tout r dans $\mathbb{R}$, la densité $g_{j+1}^\pm$ de $A^\pm((j+1)\delta, dz)$ au point r s'écrit

$$\begin{aligned} g_{j+1}^\pm(r) = \int \frac{1}{\sqrt{2\pi P((j+1)\delta)}} \exp\left\{ - \frac{\left(r - \xi^\pm_{(j+1)\delta} - z - R((j+1)\delta)z\right)^2}{2P((j+1)\delta)} \right\} F_j^\pm(dz) \\ \exp\left\{ z\varsigma^\pm_{(j+1)\delta} - \frac{1}{2}(j+1)\delta z^2 + \frac{1}{2}z^2 S((j+1)\delta) \right\}. \end{aligned} \tag{4}$$

En pratique, les $g_j^\pm(r)$ sont calculés sur une grille symétrique par rapport au point 0 et les calculs d'intégrales se font par *linéarisation* des intégrands sur la grille. Entre deux instants de discrétisation, l'algorithme n'est pas difficile à implémenter: il suffit de faire courir les systèmes auxiliaires SA_k, $k = 1, \ldots, K$ qui sont récursifs de dimension finie (ce qui prend peu de temps et de place); en revanche, à chaque instant multiple de δ, il faut calculer les valeurs de la nouvelle densité en chaque point de la grille, donc un très grand nombre d'intégrales, ce qui rend l'algorithme très lourd, déjà dans l'exemple ci-dessus pourtant le plus simple possible.

On a traité cet exemple, avec une loi initiale $F(dz)$ gaussienne centrée de variance v_0 et un bruit d'observation de variance v_b, sur ordinateur Multics. On a obtenu les temps de calcul suivants pour la densité conditionnelle à l'instant $T = 1$ correspondant à une simulation du signal et du processus observé:

(1) **2 minutes 46 sec. pour un pas de discrétisation** $\delta = 0.1$, les intégrales étant calculées **sur une grille de maille 0.1** ***entre les points*** -6 ***et*** 6;
(2) **20 minutes 27 sec. pour** $\delta = 0.01$ **avec la même grille**;
(3) **155 minutes 38 sec. pour** $\delta = 0.005$ **avec une grille deux fois plus fine**
En comparant les filtres obtenus dans ces 3 cas, il **apparaît que les résultats** du cas 1 **sont parfois assez mauvais. Il faut donc compter en fait un minimum** d'une vingtaine **de minutes de temps de calcul (pour chaque trajectoire observée) pour avoir** un résultat **fiable, et il est très net que ce temps de calcul augmente considérablement avec** la finesse **du pas de discrétisation et de la maille de la grille d'intégration**. On voit donc bien que **pour utiliser cet algorithme en dimension plus grande que** 1, il **faudrait dans** un premier **temps proposer des méthodes d'approximation pour le calcul des intégrales** de type (4).
Toujours sur cet exemple, on a pu remarquer un comportement du filtre auquel on pouvait s'attendre intuitivement: au cours du temps, les densités conditionnelles, qui demeurent symétriques par rapport à 0, restent unimodales dans certains cas et comportent deux pics lorsque le signal s'est suffisamment éloigné de 0. On a également fait varier la variance du bruit d'observation et constaté que l'apparition de deux pics est plus fréquente lorsque cette variance est petite.

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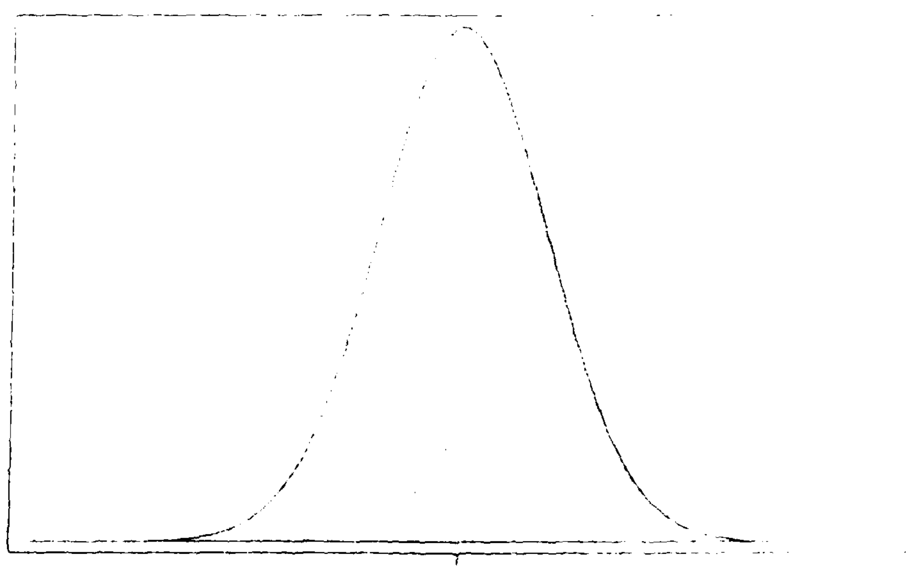
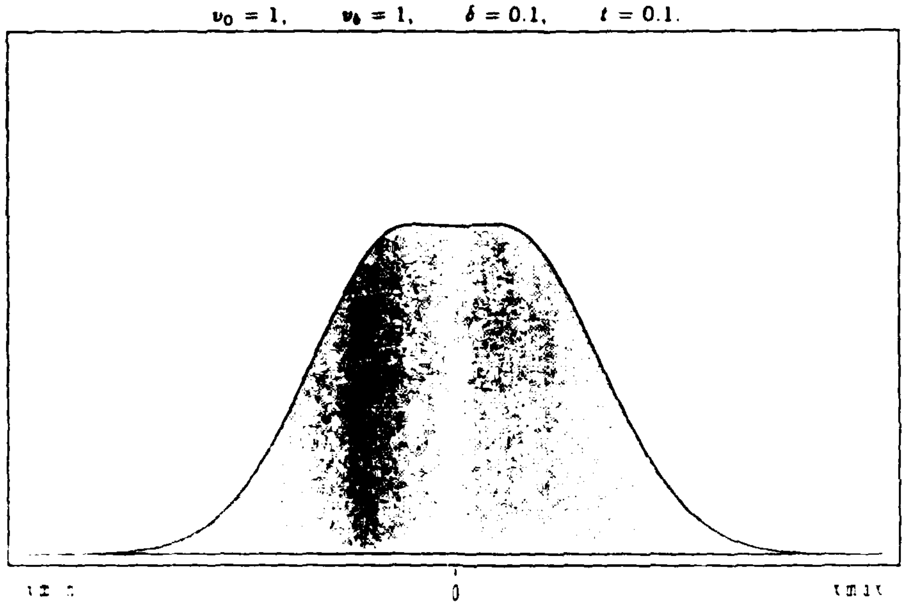
Annexe

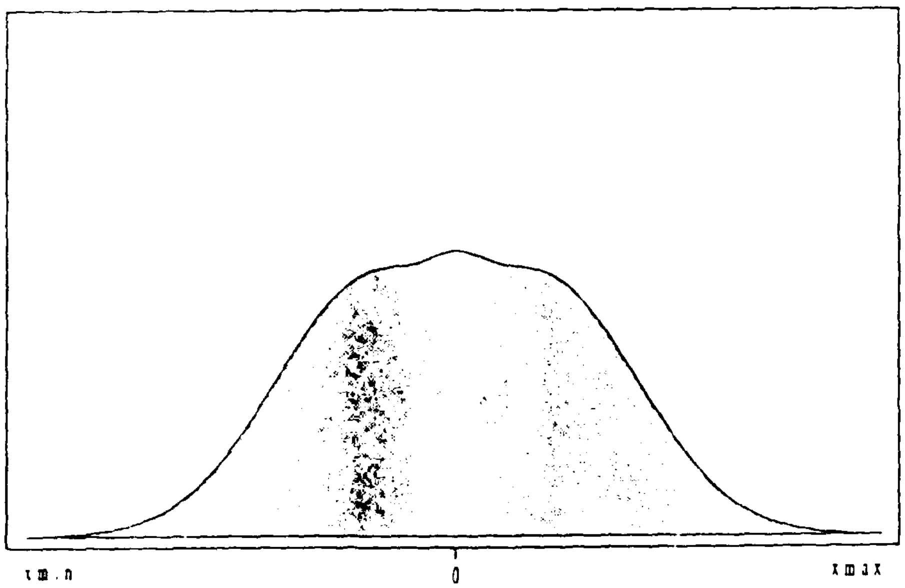
On donne ci-dessous les densités conditionnelles entre les points [illegible] instants t compris entre 0 et l'instant terminal $T = 5$ dans les deux [illegible]
1) une simulation avec une loi initiale et un bruit d'observation de variance [illegible] et un pas de discrétisation $\varepsilon = 0.1$.
2) une simulation avec une loi initiale de variance [illegible] et un bruit d'observation de variance [illegible] $= 0.1$, et un pas de discrétisation $\varepsilon =$ [illegible]



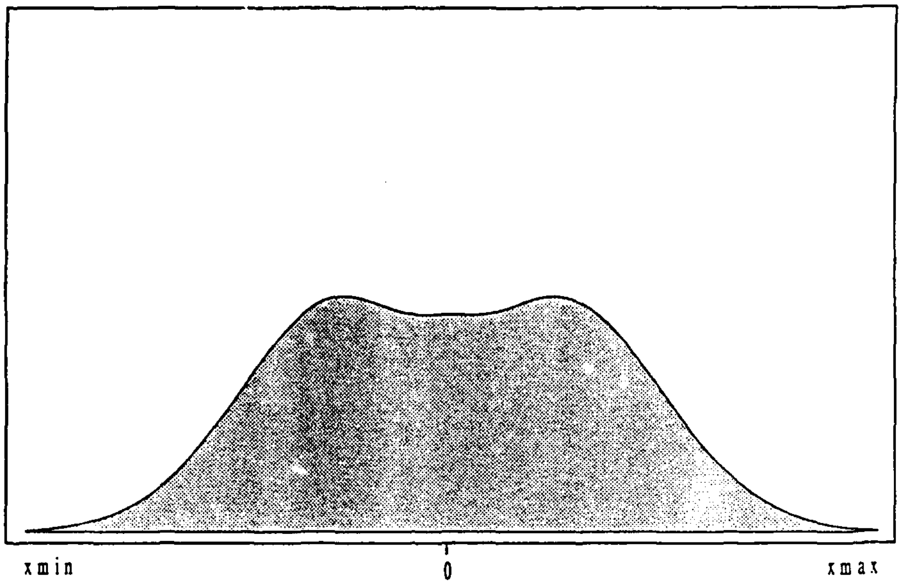
$v_0 = 1, \quad v_b = 1, \quad \delta = 0.1, \quad t = 0.1.$



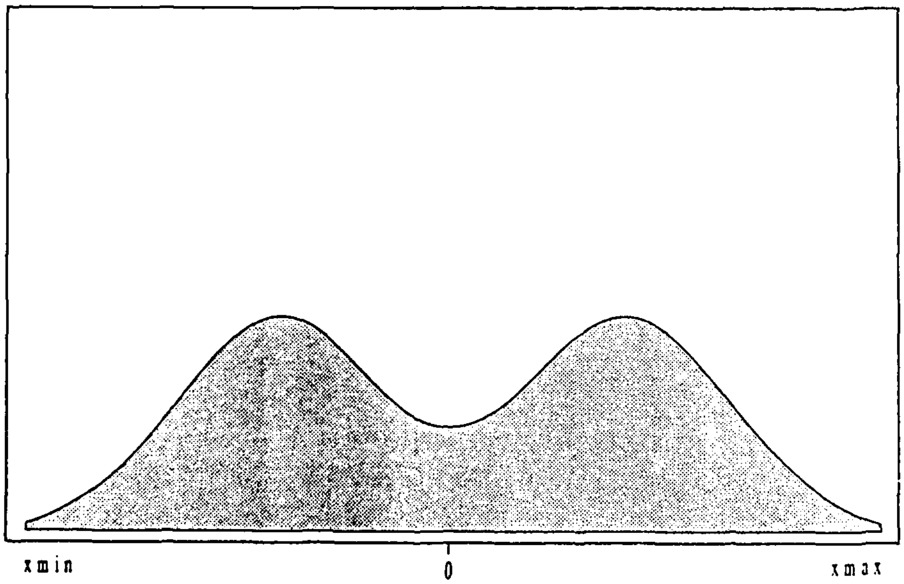
$v_0 = 1, \quad v_b = 1, \quad \delta = 0.1, \quad t = 0.3.$



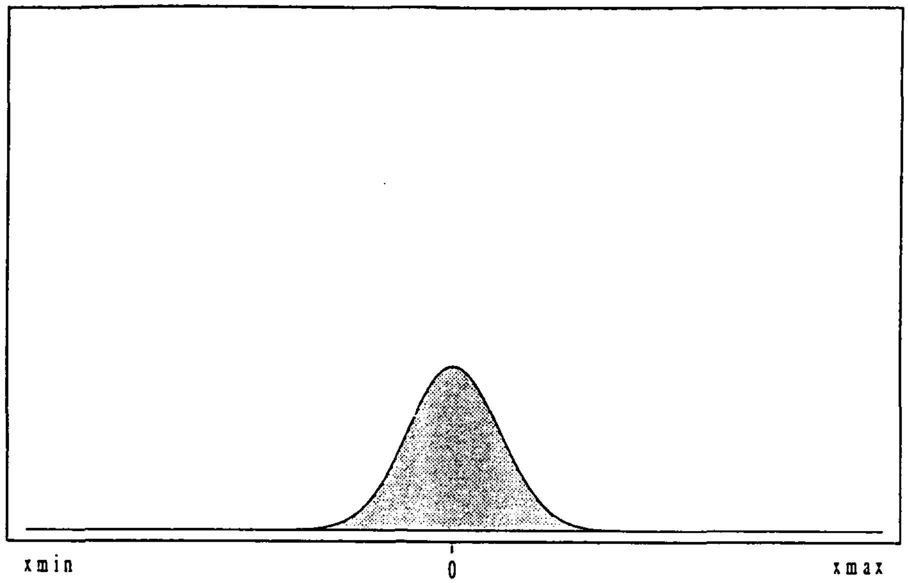
$v_0 = 1, \qquad v_b = 1, \qquad \delta = 0.1, \qquad t = 0.4.$



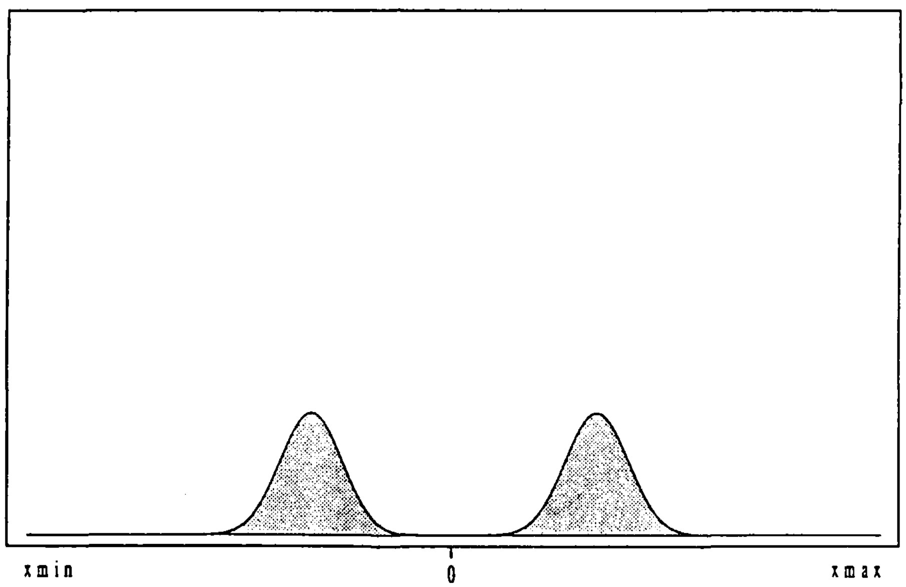
$v_0 = 1, \qquad v_b = 1, \qquad \delta = 0.1, \qquad t = 0.5.$



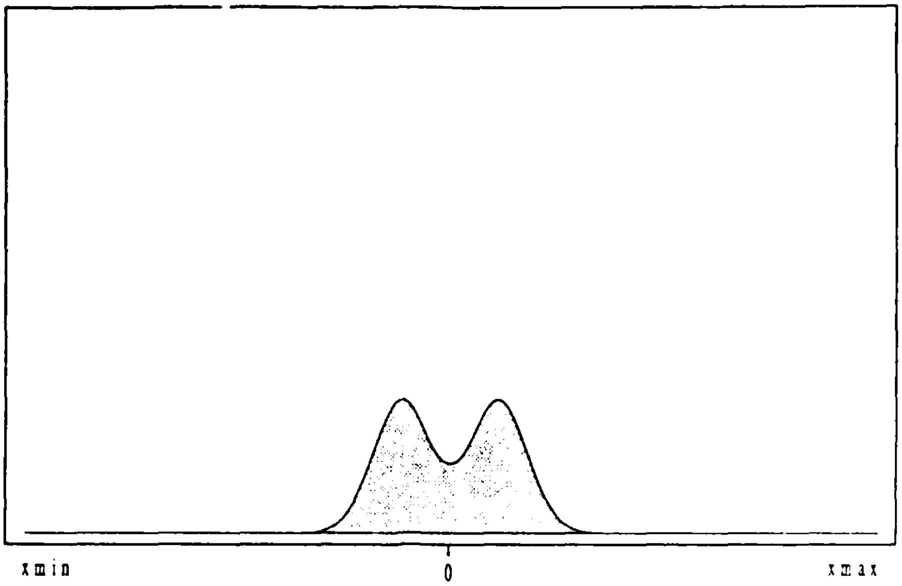
$v_0 = 0.3, \quad v_b = 0.1, \quad \delta = 0.025, \quad t = 0.$



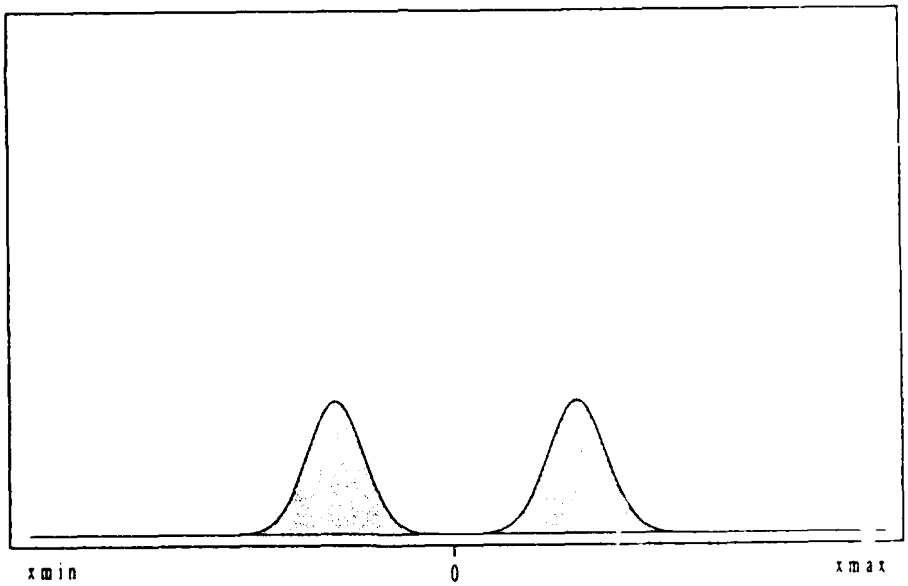
$v_0 = 0.3, \quad v_b = 0.1, \quad \delta = 0.025, \quad t = 0.1.$



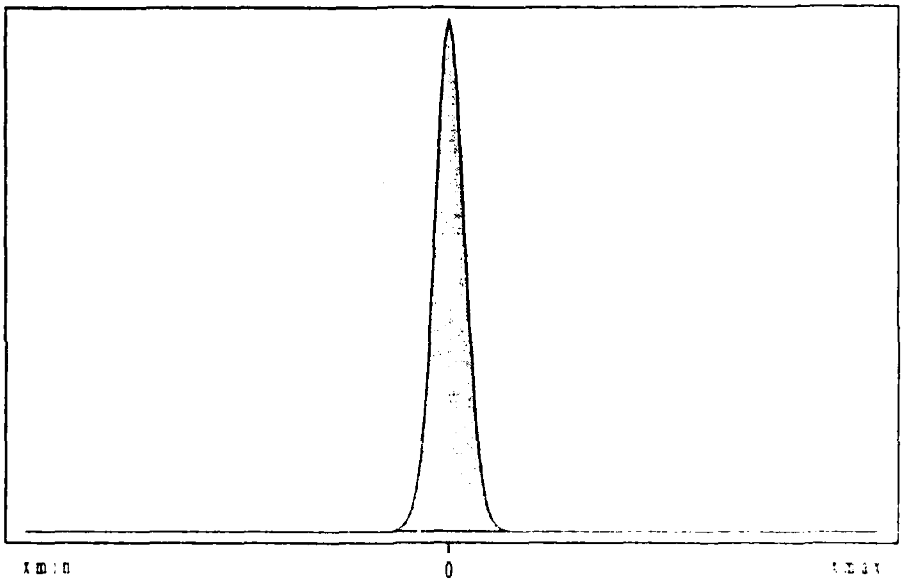
$v_0 = 0.3, \quad v_b = 0.1, \quad \delta = 0.025, \quad t = 0.375.$



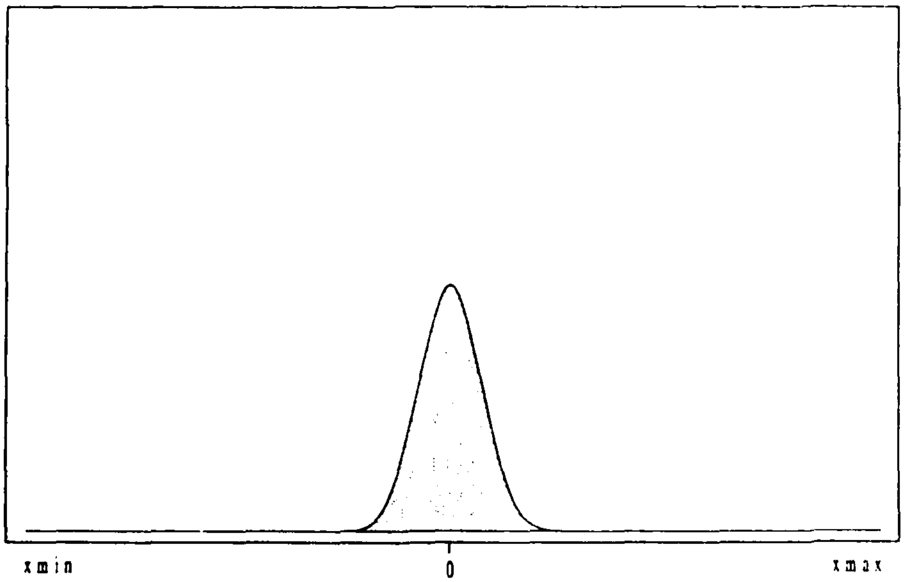
$v_0 = 0.3, \quad v_b = 0.1, \quad \delta = 0.025, \quad t = 0.425.$



$v_0 = 0.3, \quad v_b = 0.1, \quad \delta = 0.025, \quad t = 1.025.$



$v_0 = 0.3, \quad v_b = 0.1, \quad \delta = 0.025, \quad t = 1.125.$



APPENDIX 2

MLE for Partially Observed Diffusions: Direct Maximization vs. The EM Algorithm

Fabien Campillo and François Le Gland

MLE FOR PARTIALLY OBSERVED DIFFUSIONS: DIRECT MAXIMIZATION vs. THE EM ALGORITHM

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Abstract

In [1], the EM algorithm has been investigated in the context of partially observed continuous-time stochastic processes.

The purpose of this paper is to compare this approach with the direct maximization of the likelihood ratio, in the particular case of diffusion processes. This yields to a comparison of nonlinear smoothing and nonlinear filtering for the computation of a certain class of conditional expectations, relevant to the problem of estimation (Section 3). In particular, this explains why smoothing is indeed necessary for the EM algorithm approach to be efficient.

1 Introduction: the EM algorithm

The EM algorithm is an iterative method for maximizing a likelihood ratio, in a situation of partial observation [2]. Indeed, let $(P_\theta\ ;\ \theta \in \Theta)$ be a family of mutually absolutely continuous probabilities on a space $(\Omega, \mathcal{F})$, and let $\mathcal{Y} \subset \mathcal{F}$ be the σ-algebra representing all the available information. Then, the log-likelihood ratio can be defined as:

$$L(\theta) \triangleq \log \mathbf{E}_\alpha \left(\frac{dP_\theta}{dP_\alpha} \mid \mathcal{Y} \right)$$

where α is fixed in Θ, and the MLE (maximum likelihood estimate) as:

$$\widehat{\theta} \in \arg\max_{\theta\in\Theta} L(\theta)$$

The EM algorithm is based on the following direct application of Jensen's inequality:

$$L(\theta) - L(\theta') = \log \mathbf{E}_{\theta'} \left(\frac{dP_\theta}{dP_{\theta'}} \mid \mathcal{Y} \right) \geq \mathbf{E}_{\theta'} \left(\log \frac{dP_\theta}{dP_{\theta'}} \mid \mathcal{Y} \right) \triangleq Q(\theta, \theta') \tag{1}$$

which gives, for each value θ' of the parameter, a minoration of the log-likelihood function $\theta \longrightarrow L(\theta)$ by means of an auxiliary function $\theta \longrightarrow L(\theta') + Q(\theta, \theta')$, with equality at $\theta = \theta'$.

The way the EM algorithm works is described by the flow chart given in Fig. 2, whereas Fig. 1 shows a sample few steps of the algorithm.

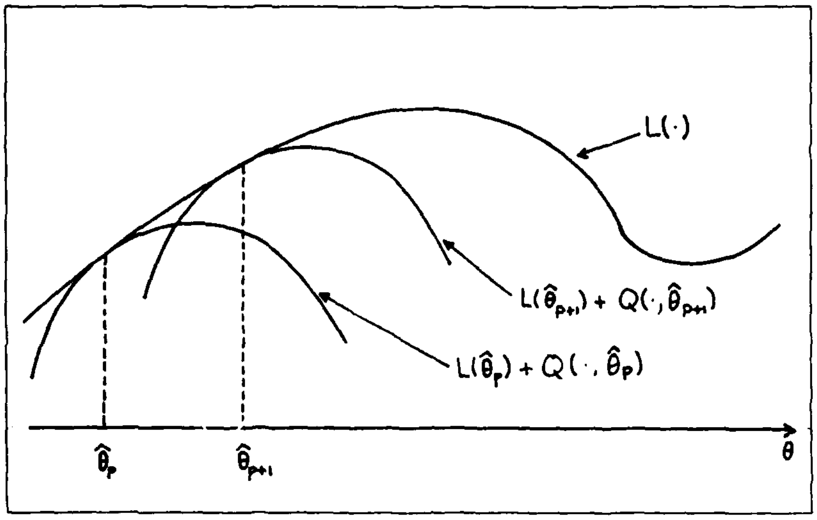

Figure 1: A sample iteration of the algorithm

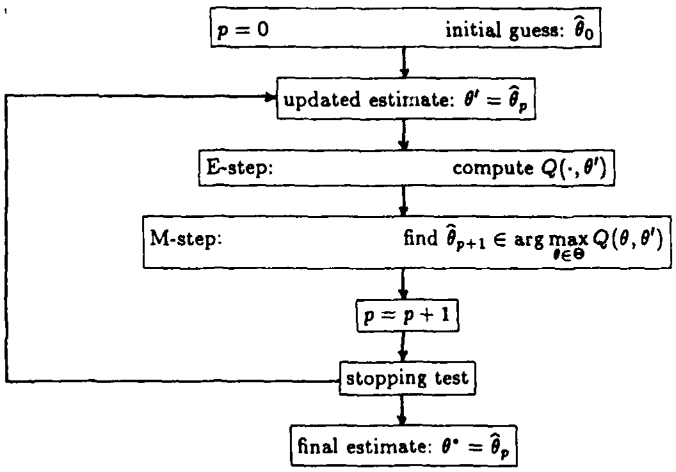

Figure 2: Algorithm flow chart

An interesting feature of the algorithm is that it generates a maximizing sequence $\{\widehat{\theta}_p \; ; \; p = 0,1,\cdots\}$ in the sense that: $L(\widehat{\theta}_{p+1}) \geq L(\widehat{\theta}_p)$. Some general convergence results about the sequences $\{L(\widehat{\theta}_p) \; ; \; p = 0,1,\cdots\}$ or $\{\widehat{\theta}_p \; ; \; p = 0,1,\cdots\}$ are proved in [10], under mild regularity assumptions on $L(\cdot)$ and $Q(\cdot,\cdot)$.

For this algorithm to be interesting from a computational point of view, the following two features should be found:

(E) computing the auxiliary function $Q(\cdot,\theta')$ should not be much more complicated than computing the original log-likelihood ratio $L(\cdot)$,

(M) maximizing the auxiliary function $Q(\cdot,\theta')$ should be quite simpler than maximizing the original log-likelihood ratio $L(\cdot)$.

The latter will occur if $Q(\theta,\theta')$ – as could be expected from the definition (1) – can be explicitely computed by means of a (generally infinite-dimensional) density depending only on θ', acting on various simple functions depending on both θ and θ'. If this is the case, computing $Q(\theta,\theta')$ or the gradient $\nabla^{10}Q(\theta,\theta')$ with respect to θ, for different values of the parameter θ (θ' being fixed), will not involve the computation of any other infinite-dimensional object.

To prove the existence of smooth enough – in the a.s. sense – versions of $\theta \mapsto L(\theta)$ and $(\theta,\theta') \mapsto Q(\theta,\theta')$, as well as to get the expression of the corresponding derivatives, one can rely on the following extension of Kolmogorov's lemma, and the next remark:

Proposition 1.1 [9, Lemma 1]

Let $(\Omega,\mathcal{F},P)$ be a probability space and $(A(\theta) \; ; \; \theta \in \Theta)$, with $\Theta \subset \mathbf{R}^p$, such that:

$\theta \mapsto A(\theta)$ is of class $C^{k,a}$ (i.e. k-times continuously differentiable with its k-th derivative Hölder-continuous of order $0 \leq a \leq 1$) from Θ to $L^r(\Omega,\mathcal{F},P)$

Then there exists a random function $(\theta,\omega) \mapsto \widetilde{A}(\theta,\omega)$ such that:

- *$\forall\omega \in \Omega \; ; \; \theta \mapsto \widetilde{A}(\theta,\omega)$ is of class C^j provided $j + \dfrac{p}{r} < k + a$*
- *$\forall\theta \in \Theta \; ; \; \widetilde{A}(\theta,\cdot)$ is $\mathcal{F}$-measurable, and the a.s. derivatives of $\widetilde{A}(\theta,\cdot)$ (up to order j) are a.s. equal to the corresponding L^r-derivatives of $A(\theta)$*

Remark: Let $\mathcal{Y} \subset \mathcal{F}$ be a sub σ-algebra. To prove the existence of an a.s. smooth version of $\theta \mapsto B(\theta)$ with $B(\theta) \triangleq \mathrm{E}(A(\theta) \mid \mathcal{Y})$, it is enough to check that $\theta \mapsto A(\theta)$ satisfies the assumptions of the previous Proposition. Moreover the a.s. derivatives of (the smooth version of) $B(\theta)$ will be a.s. equal to the conditional expectations with respect to $\mathcal{Y}$ of the corresponding derivatives of (the smooth version of) $A(\theta)$.

The EM algorithm has been applied in the context of continuous-time stochastic processes in [1] where, in the case of diffusion processes [1, Section 3], the general expression of $Q(\theta,\theta')$ has been derived [1, (3.4)] and said to involve a nonlinear smoothing problem. The authors have also considered some particular cases in order to get more tractable results, as well as other situations including finite-state Markov processess and linear systems [1, Sections 4–5].

The purpose of this paper is to get back to the general problem for diffusion processes and address the following three points:

- clarify the expression [1, (3.4)] giving $Q(\theta, \theta')$ in terms of a nonlinear smoothing problem,
- get an equivalent expression for $Q(\theta, \theta')$ and its gradient $\nabla^{10} Q(\theta, \theta')$, in terms of a nonlinear filtering problem (it will turn out that smoothing is indeed necessary for the point [M] introduced above to be satisfied, although filtering is enough to compute $Q(\theta, \theta')$ for a given value of (θ, θ')),
- get similar expressions for the original log-likelihood ratio $L(\theta)$ and its gradient $\nabla L(\theta)$.

This will allow to compare, from a computational point of view, the two possible approaches for maximum likelihood estimation:

- direct maximization of the likelihood ratio,
- the EM algorithm.

Finally, it should be mentionned that the scope of this paper is limited to "exact" formulas, in terms of stochastic PDE's (or their discretized approximations).

2 Statistical framework

In this section, expressions for the log-likelihood ratio $L(\cdot)$ and the auxiliary function $Q(\cdot, \cdot)$ will be derived in the following context (see [1, Section 3]).

Hypotheses:

Let $\theta \in \Theta \subset \mathbf{R}^p$ denote the unknown parameter. Assume:

- $(p_0^\theta(\cdot) \; ; \; \theta \in \Theta)$ are mutually absolutely continuous densities on $\mathbf{R}^m$,
- $b_\theta(\cdot)$ is a measurable and bounded function from $\mathbf{R}^m$ to $\mathbf{R}^m$,
- $\sigma(\cdot)$ is a continuous and bounded function on $\mathbf{R}^m$ such that $a(\cdot) \stackrel{\Delta}{=} \sigma(\cdot)\sigma^*(\cdot)$ is a uniformly strictly elliptic $m \times m$ matrix, i.e. $a(\cdot) \geq \alpha I$, and $\sum_{i=1}^{m} \frac{\partial}{\partial x_i} a^{ij}(\cdot)$ is a measurable and bounded function from $\mathbf{R}^m$ to $\mathbf{R}^m$, for: $j = 1, \ldots, m$,
- $h_\theta(\cdot)$ is a measurable and bounded function from $\mathbf{R}^m$ to $\mathbf{R}^d$.

Additional hypotheses concerning the regularity with respect to the parameter θ will be needed later on.

Suppose then that a family $(P_\theta \; ; \; \theta \in \Theta)$ of probabilities is given on a space $(\Omega, \mathcal{F})$, together with a pair of stochastic processes $(X_t \; ; \; 0 \leq t \leq T)$ and $(Y_t \; ; \; 0 \leq t \leq T)$ taking values in $\mathbf{R}^m$ and $\mathbf{R}^d$ respectively, such that under P_θ:

$$
\begin{aligned}
dX_t &= b_\theta(X_t)\, dt + \sigma(X_t)\, dW_t^\theta \qquad\qquad X_0 \sim p_0^\theta(\cdot) \\
dY_t &= h_\theta(X_t)\, dt + d\overline{W}_t^\theta
\end{aligned}
$$

where $(W_t^\theta\ ;\ 0 \le t \le T)$ and $(\overline{W}_t^\theta\ ;\ 0 \le t \le T)$ are independent Wiener processes, and the initial condition X_0 is a r.v. independent of both. Then $(P_\theta\ ;\ \theta \in \Theta)$ are mutually absolutely continuous probabilities on $(\Omega, \mathcal{F})$ with:

$$\begin{aligned}
\Lambda_{\theta\theta'} &\triangleq \frac{dP_\theta}{dP_{\theta'}} \\
&= \frac{p_0^\theta(X_0)}{p_0^{\theta'}(X_0)} \exp\left\{\int_0^T (a^{-1}(X_s)(b_\theta(X_s) - b_{\theta'}(X_s)))^* \sigma(X_s)\, dW_s^{\theta'}\right. \\
&\quad \left. -\frac{1}{2}\int_0^T (b_\theta(X_s) - b_{\theta'}(X_s))^* a^{-1}(X_s)(b_\theta(X_s) - b_{\theta'}(X_s))\, ds\right\} \qquad (2) \\
&\quad \exp\left\{\int_0^T (h_\theta(X_s) - h_{\theta'}(X_s))^*\, dY_s - \frac{1}{2}\int_0^T (h_\theta^*(X_s)h_\theta(X_s) - h_{\theta'}^*(X_s)h_{\theta'}(X_s))\, ds\right\}
\end{aligned}$$

Consider also the probability $\overset{\circ}{P}_\theta$ defined by:

$$Z^\theta \triangleq \frac{dP_\theta}{d\overset{\circ}{P}_\theta} = \exp\left\{\int_0^T h_\theta^*(X_s)\, dY_s - \frac{1}{2}\int_0^T h_\theta^*(X_s)h_\theta(X_s)\, ds\right\}$$

so that, under $\overset{\circ}{P}_\theta$:

$$dX_t = b_\theta(X_t)\, dt + \sigma(X_t)\, dW_t^\theta \qquad\qquad X_0 \sim p_0^\theta(\cdot)$$

and $(Y_t\ ;\ 0 \le t \le T)$ is a Wiener processes independent of $(W_t^\theta\ ;\ 0 \le t \le T)$, and the r.v. X_0 is again independent of both. $\Lambda_{\theta\theta'}$ can then be decomposed as:

$$\Lambda_{\theta\theta'} = U_{\theta\theta'} \frac{Z^\theta}{Z^{\theta'}} \qquad \text{with:} \qquad U_{\theta\theta'} \triangleq \frac{d\overset{\circ}{P}_\theta}{d\overset{\circ}{P}_{\theta'}}$$

It is assumed that only $(Y_t\ ;\ 0 \le t \le T)$ is observed, and let $(\mathcal{Y}_t\ ;\ 0 \le t \le T)$ denote the associated filtration. Then the likelihood ratio for the estimation of the parameter θ can be expressed as:

$$\overset{\circ}{\mathbf{E}}_\alpha\left(\frac{dP_\theta}{d\overset{\circ}{P}_\alpha} \mid \mathcal{Y}_T\right) = \overset{\circ}{\mathbf{E}}_\alpha\left(Z^\theta U_{\theta\alpha} \mid \mathcal{Y}_T\right)$$

where α is fixed in Θ. By Bayes formula:

$$\overset{\circ}{\mathbf{E}}_\alpha\left(Z^\theta U_{\theta\alpha} \mid \mathcal{Y}_T\right) = \overset{\circ}{\mathbf{E}}_\theta\left(Z^\theta \mid \mathcal{Y}_T\right) \times \overset{\circ}{\mathbf{E}}_\alpha\left(U_{\theta\alpha} \mid \mathcal{Y}_T\right) = \overset{\circ}{\mathbf{E}}_\theta\left(Z^\theta \mid \mathcal{Y}_T\right)$$

since $U_{\theta\alpha}$ is independent of $\mathcal{Y}_T$ under $\overset{\circ}{P}_\alpha$.

This gives the following two expressions for the log-likelihood ratio $L(\cdot)$:

$$L(\theta) = \log \overset{\circ}{\mathbf{E}}_\alpha\left(Z^\theta U_{\theta\alpha} \mid \mathcal{Y}_T\right) \qquad (3)$$

$$= \log \overset{\circ}{\mathbf{E}}_\theta\left(Z^\theta \mid \mathcal{Y}_T\right) \qquad (4)$$

For the auxiliary function $Q(\cdot,\cdot)$ defined by (1), one has immediately:

$$Q(\theta,\theta') = \mathbf{E}_{\theta'}(\log \Lambda_{\theta\theta'} \mid \mathcal{Y}) \tag{5}$$

$$= \frac{\mathring{\mathbf{E}}_{\theta'}\left(Z^{\theta'} \log \Lambda_{\theta\theta'} \mid \mathcal{Y}_T\right)}{\mathring{\mathbf{E}}_{\theta'}\left(Z^{\theta'} \mid \mathcal{Y}_T\right)} \tag{6}$$

$$= \frac{\mathring{\mathbf{E}}_{\alpha}\left(Z^{\theta'} U_{\theta'\alpha} \log \Lambda_{\theta\theta'} \mid \mathcal{Y}_T\right)}{\mathring{\mathbf{E}}_{\alpha}\left(Z^{\theta'} U_{\theta'\alpha} \mid \mathcal{Y}_T\right)} \tag{7}$$

Remark: Formulas (4) and (6) will be used to compute the log-likelihood ratio and the auxiliary function respectively by means of a nonlinear filtering problem, formula (5) directly allow to compute the auxiliary function by means of a nonlinear smoothing problem, whereas formulas (3) and (7) should be used to prove the existence of smooth versions and get the expression of the corresponding derivatives.

Indeed, under additional regularity asumptions, it is easy to prove, using Proposition 1.1, that both $\theta \mapsto L(\theta)$ and $\theta \to Q(\theta,\theta')$ have a.s. differentiable versions, with gradients given by:

$$\nabla L(\theta) = \frac{\mathring{\mathbf{E}}_{\theta}\left(\rho^{\theta} Z^{\theta} \mid \mathcal{Y}_T\right)}{\mathring{\mathbf{E}}_{\theta}\left(Z^{\theta} \mid \mathcal{Y}_T\right)} = \mathbf{E}_{\theta}(\rho^{\theta} \mid \mathcal{Y}_T) \tag{8}$$

$$\nabla^{10} Q(\theta,\theta') = \mathbf{E}_{\theta'}(\rho^{\theta} \mid \mathcal{Y}_T) = \frac{\mathring{\mathbf{E}}_{\theta'}\left(\rho^{\theta} Z^{\theta} \mid \mathcal{Y}_T\right)}{\mathring{\mathbf{E}}_{\theta'}\left(Z^{\theta} \mid \mathcal{Y}_T\right)} \tag{9}$$

respectively, where (∇ denoting derivation with respect to the parameter θ):

$$\rho^{\theta} \triangleq \frac{\nabla p_0^{\theta}(X_0)}{p_0^{\theta}(X_0)} + \int_0^T (a^{-1}(X_s)\nabla b_{\theta}(X_s))^{*}\sigma(X_s)\, dW_s^{\theta} + \int_0^T (\nabla h_{\theta}(X_s))^{*}\,(dY_s - h_{\theta}(X_s)\, ds) \tag{10}$$

Remark: One can check from (8) and (9) that:

$$\nabla^{10} Q(\theta,\theta') \mid_{\theta=\theta'} = \nabla L(\theta')$$

as expected.

The next section will be devoted to give different ways, by means of SPDE mainly, to compute the various quantities introduced so far: $L(\theta)$, $\nabla L(\theta)$, $Q(\theta,\theta')$ and $\nabla^{10} Q(\theta,\theta')$. This will make possible the numerical implementation of algorithms for the maximization of the likelihood ratio.

3 Smoothing vs. filtering for the computation of a certain class of conditional expectations

For the sake of simplicity, any reference to the parameter θ will be dropped throughout this section. In particular, P will denote the probability under which:

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t \qquad X_0 \sim p_0(\cdot)$$

$$dY_t = h(X_t)\,dt + d\overline{W}_t$$

where $(W_t\ ;\ 0 \le t \le T)$ and $(\overline{W}_t\ ;\ 0 \le t \le T)$ are independent Wiener processes, whereas under $\overset{\circ}{P}$:

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t \qquad\qquad X_0 \sim p_0(\cdot)$$

and $(Y_t\ ;\ 0 \le t \le T)$ is a Wiener processes independent of $(W_t\ ;\ 0 \le t \le T)$. Define also the process $(Z_t\ ;\ 0 \le t \le T)$ by:

$$Z_t = \exp\left\{\int_0^t h^*(X_s)\,dY_s - \frac{1}{2}\int_0^t h^*(X_s)h(X_s)\,ds\right\}$$

The purpose of this section is to provide two different ways - one based on nonlinear smoothing, the other on nonlinear filtering - for the computation of the following class of conditional expectations:

$$A \triangleq \mathbf{E}\left(\beta(X_0) + \int_0^T \xi(X_s)\,ds + \int_0^T \eta^*(X_s)\,dY_s + \int_0^T \chi^*(X_s)\sigma(X_s)\,dW_s \;\Big|\; \mathcal{Y}_T\right) \qquad (11)$$

where β, ξ, η and χ are measurable and bounded functions from $\mathbf{R}^m$ to $\mathbf{R}$, $\mathbf{R}$, $\mathbf{R}^d$ and $\mathbf{R}^m$ respectively. It is readily seen that the computation of either $\nabla L(\theta)$, $Q(\theta,\theta')$ or $\nabla^{10}Q(\theta,\theta')$ involves such conditional expectations.

It is clear from the definition that A depends linearly on (β,ξ,η,χ). It will turn out that nonlinear smoothing is the only way to make this dependence explicit, although nonlinear filtering – which is simpler – is enough to just compute A. The following facts and notations about nonlinear filtering and smoothing equations are gathered here, and will be extensively used in the sequel:

Notations:

• Filtering

π_t (resp. u_t) will always denote the unnormalized (resp normalized) conditional density of the r.v. X_t given $\mathcal{Y}_t$, i.e.:

$$(\pi_t,\phi) \triangleq \mathbf{E}(\phi(X_t)\mid \mathcal{Y}_t)$$

$$(u_t,\phi) \triangleq \overset{\circ}{\mathbf{E}}\,(\phi(X_t)Z_t \mid \mathcal{Y}_t) \qquad (12)$$

where ϕ is a test-function. By Bayes formula:

$$(\pi_t,\phi) = \frac{(u_t,\phi)}{(u_t,1)} \qquad (13)$$

The equation for $(u_t\ ;\ 0 \le t \le T)$ is Zakaï equation [4]:

$$du_t = \mathcal{L}^* u_t\,dt + h^* u_t\,dY_t \qquad\qquad u_0 = p_0 \qquad (14)$$

where $\mathcal{L}^*$ denotes the adjoint operator of the generator of the diffusion process $(X_t\ ;\ 0 \le t \le T)$, i.e.:

$$\mathcal{L} \triangleq \frac{1}{2}\sum_{i,j=1}^m a^{ij}(\cdot)\frac{\partial^2}{\partial x_i\partial x_j} + \sum_{i=1}^m b^i(\cdot)\frac{\partial}{\partial x_i}$$

• Smoothing (fixed-interval)

Let $T > 0$ denote the fixed end time. π_t (resp. q_t) will always denote the unnormalized (resp. normalized) conditional density of the r.v. X_t given $\mathcal{Y}_T$, i.e.

$$(\pi_t, \phi) \doteq \mathbf{E}(\phi(X_t) \mid \mathcal{Y}_T)$$

$$(q_t, \phi) \doteq \mathring{\mathbf{E}}(\phi(X_t) Z_T \mid \mathcal{Y}_T)$$

Again

$$(\pi_t, \phi) = \frac{(q_t, \phi)}{(q_t, 1)}$$

Introducing the backward Zakai equation

$$dv_t + \mathcal{L} v_t \, dt + h^* v_t \, dY_t = 0 \quad [illegible] \quad v_T = 1$$

one has [4]

$$[illegible] \text{ is independent of } t$$

Let $(U_t^s, 0 \le s \le t \le T)$ (resp. [illegible]) denote the forward (resp. backward) stochastic semi-group associated with equation (4) (resp. (6)) (see [illegible] for the definition of stochastic semi-groups). Then (7) can be restated as

[illegible]

The next results are proved in [illegible]

[illegible]

with the differential

[illegible]

This gives a couple of equations for [illegible]

(backward) [illegible]

(forward) [illegible]

Moreover, it follows from (7) and [illegible] that

$$[illegible] \text{ is independent of } t$$

From the computational point of view [illegible] store the value of the unnormalized [illegible] the backward equation for q_t [illegible] evaluate [illegible] capacity of memory storage.

The most direct approach to compute [illegible] nonlinear smoothing [illegible] was used in [illegible]

3.1 Smoothing

This approach gives (see [1, (3.4)]):

$$
\begin{aligned}
A &= \mathbf{E}(\beta(X_0) \mid \mathcal{Y}_T) + \int_0^T \mathbf{E}(\xi(X_s) \mid \mathcal{Y}_T)\, ds + \int_0^T \mathbf{E}(\eta^*(X_s) \mid \mathcal{Y}_T)\, dY_s \\
&\quad + \mathbf{E}\left(\int_0^T \chi^*(X_s)\sigma(X_s)\, dW_s \mid \mathcal{Y}_T\right)
\end{aligned} \tag{23}
$$

The mathematical meaning of the third term is not clear, since the integrand is obviously not adapted to the filtration $(\mathcal{Y}_t \,;\, 0 \leq t \leq T)$. However:

- in discrete-time, the situation would be quite easy to understand,
- a rigorous meaning can be given indeed, in terms of the two-sided stochastic integral introduced in [6], [7]: this will appear as a by-product of the next approach, based on nonlinear filtering.

On the other hand, there is still no computable expression available for the last term. The only situations where [1] gives such an explicit expression, can indeed be considered as particular cases of the following

Lemma 3.1

Assume there exists a scalar function $F \in C_b^2(\mathbf{R}^m)$ such that:

$$\chi = DF \tag{24}$$

where D denotes the derivative with respect to the space variable. Then:

$$\mathbf{E}\left(\int_0^T \chi^*(X_s)\sigma(X_s)\, dW_s \mid \mathcal{Y}_T\right) = \mathbf{E}(F(X_T) \mid \mathcal{Y}_T) - \mathbf{E}(F(X_0) \mid \mathcal{Y}_T) - \int_0^T \mathbf{E}(\mathcal{L}F(X_s) \mid \mathcal{Y}_T)\, ds \tag{25}$$

whose proof follows immediately from Itô's lemma. In the general case, a computable expression will be obtained as a consequence of the approach based on nonlinear filtering, see (29) below.

With the notations introduced at the beginning of this section, one has under assumption (24)

$$A = (\bar{\pi}_0, \beta) + \int_0^T (\bar{\pi}_s, \xi)\, ds + \int_0^T (\bar{\pi}_s, \eta^*)\, dY_s + (\bar{\pi}_T, F) - (\bar{\pi}_0, F) - \int_0^T (\bar{\pi}_s, \mathcal{L}F)\, ds \tag{26}$$

where the exact meaning of the stochastic integral is still not precised.

3.2 Filtering

Define

$$\rho_t \doteq \beta(X_0) + \int_0^t \xi(X_s)\, ds + \int_0^t \eta^*(X_s)\, dY_s + \int_0^t \chi^*(X_s)\sigma(X_s)\, dW_s$$

so that, by Bayes formula:

$$A = \mathbf{E}(\rho_T \mid \mathcal{Y}_T) = \frac{\overset{\circ}{\mathbf{E}}\,(\rho_T Z_T \mid \mathcal{Y}_T)}{\overset{\circ}{\mathbf{E}}\,(Z_T \mid \mathcal{Y}_T)}$$

The idea is to find an equation for $(w_t\ ;\ 0 \le t \le T)$ defined by:

$$(w_t, \phi) \stackrel{\Delta}{=} \overset{\circ}{\mathbf{E}}\,(\phi(X_t)\rho_t Z_t \mid \mathcal{Y}_t)$$

By Itô's lemma:

$$d[\phi(X_t)\rho_t Z_t] = \rho_t Z_t \mathcal{L}\phi(X_t)\,dt + \rho_t Z_t (D\phi(X_t))^* \sigma(X_t)\,dW_t$$

$$+\phi(X_t) Z_t \xi(X_t)\,dt + \phi(X_t) Z_t \eta^*(X_t)\,dY_t + \phi(X_t) Z_t \chi^*(X_t)\sigma(X_t)\,dW_t$$

$$+\phi(X_t)\rho_t h^*(X_t) Z_t\,dY_t + \phi(X_t)\eta^*(X_t) h(X_t) Z_t\,dt + Z_t (D\phi(X_t))^* a(X_t)\chi(X_t)\,dt$$

Using known properties of conditional expectation given the observation under the reference probability $\overset{\circ}{P}$, and the definition (12), one gets:

$$(w_t, \phi) = (p_0, \beta\phi) + \int_0^t (w_s, \mathcal{L}\phi)\,ds + \int_0^t (w_s, h^*\phi)\,dY_s$$

$$+\int_0^t (u_s, \xi)\,ds + \int_0^t (u_s, \eta^*)\,dY_s + \int_0^t (u_s, \eta^* h\phi)\,ds + \int_0^t (u_s, \mathcal{J}(\chi)\phi)\,ds$$

where:

$$\mathcal{J}(\chi)\phi \stackrel{\Delta}{=} \chi^* a D\phi$$

The equation satisfied (at least in a weak sense) by $(w_t\ ;\ 0 \le t \le T)$ is therefore:

$$\begin{aligned} dw_t &= \mathcal{L}^* w_t\,dt + h^* w_t\,dY_t + (\xi + \eta^* h)u_t\,dt + \eta^* u_t\,dY_t + \mathcal{J}^*(\chi)u_t\,dt \\ w_0 &= \beta p_0 \end{aligned} \tag{27}$$

With the notations introduced above, one has:

$$A = \frac{(w_T, 1)}{(u_T, 1)} \tag{28}$$

This expression is obviously simpler, and cheaper to compute, than the corresponding equation (26) obtained by smoothing. Unfortunately, the linear dependence of $(w_T, 1)$ on (β, ξ, η, χ) is not made explicit, which should be the case for the point [M] to be satisfied. Therefore, the next step will be to make this dependence more explicit. Basically, one will recover the solution based on smoothing, so that there seems to be little gain overall. However, there will be some benefit:

- the stochastic integral in (23) will be given a rigorous meaning,
- the last term in (23) will also be given a computable expression, whether or not assumption (24) is satisfied.

3.3 Back to smoothing

Because of linearity, the following decomposition holds (with obvious notations):

$$A = A^{(0)}(\beta) + A^{(1)}(\xi) + A^{(2)}(\eta) + A^{(3)}(\chi)$$

For each term of the decomposition, there exists a representation such as (28), in terms of the solution of a SPDE. These will be studied separately, using a "variation of constant" argument involving the stochastic semi-groups introduced above.

- **Study of $A^{(0)}(\beta)$**

$$A^{(0)}(\beta) = \frac{(w_T^{(0)}, 1)}{(u_T, 1)}$$

where:

$$dw_t^{(0)} = \mathcal{L}^* w_t^{(0)}\, dt + h^* w_t^{(0)}\, dY_t \qquad\qquad w_0^{(0)} = \beta p_0$$

This gives successively, using in particular (18):

$$\begin{aligned} w_t^{(0)} &= U_t^0[\beta p_0] \\ (w_t^{(0)}, \phi) &= (U_t^0[\beta p_0], \phi) = (\beta p_0, V_t^0 \phi) \\ (w_T^{(0)}, 1) &= (\beta p_0, v_0) = (q_0, \beta) \end{aligned}$$

Finally, using (22):

$$A^{(0)}(\beta) = \frac{(q_0, \beta)}{(u_T, 1)} = (\overline{\pi}_0, \beta)$$

which is exactly the first term in (26).

- **Study of $A^{(1)}(\xi)$**

$$A^{(1)}(\xi) = \frac{(w_T^{(1)}, 1)}{(u_T, 1)}$$

where:

$$dw_t^{(1)} = \mathcal{L}^* w_t^{(1)}\, dt + h^* w_t^{(1)}\, dY_t + \xi u_t\, dt \qquad\qquad w_0^{(1)} = 0$$

This gives successively, using in particular (18):

$$\begin{aligned} w_t^{(1)} &= \int_0^t U_t^s[\xi u_s]\, ds \\ (w_t^{(1)}, \phi) &= \int_0^t (U_t^s[\xi u_s], \phi)\, ds = \int_0^t (\xi u_s, V_t^s \phi)\, ds \\ (w_T^{(1)}, 1) &= \int_0^T (\xi u_s, v_s)\, ds = \int_0^T (q_s, \xi)\, ds \end{aligned}$$

Finally, using (22):

$$A^{(1)}(\xi) = \frac{\int_0^T (q_s, \xi)\, ds}{(u_T, 1)} = \int_0^T (\overline{\pi}_s, \xi)\, ds$$

which is exactly the second term in (26).

- **Study of $A^{(3)}(\chi)$**

$$A^{(3)}(\chi) = \frac{(w_T^{(3)}, 1)}{(u_T, 1)}$$

where:

$$dw_t^{(3)} = \mathcal{L}^* w_t^{(3)}\, dt + h^* w_t^{(3)}\, dY_t + J^*(\chi) u_t\, dt \qquad\qquad w_0^{(3)} = 0$$

This gives successively, using again (18):

$$\begin{aligned}
w_t^{(3)} &= \int_0^t U_t^s [J^*(\chi) u_s]\, ds \\
(w_t^{(3)}, \phi) &= \int_0^t (U_t^s [J^*(\chi) u_s], \phi)\, ds \\
&= \int_0^t (J^*(\chi) u_s, V_t^s \phi)\, ds = \int_0^t (u_s, J(\chi)[V_t^s \phi])\, ds \\
(w_T^{(3)}, 1) &= \int_0^T (u_s, J(\chi) v_s)\, ds \\
&= \int_0^T (u_s, \chi^* a D v_s)\, ds = \int_0^T (q_s, \chi^* a \frac{D v_s}{v_s})\, ds
\end{aligned}$$

From the identity:

$$u_s D v_s = (q_s, 1)\, \pi_s\, D \left[\frac{\overline{\pi}_s}{\pi_s}\right]$$

one finally gets, using again (22):

$$A^{(3)}(\chi) = \frac{\int_0^T (u_s, \chi^* a D v_s)\, ds}{(u_T, 1)} = \int_0^T (\chi^* a D \left[\frac{\overline{\pi}_s}{\pi_s}\right], \pi_s)\, ds \tag{29}$$

The link with the partial result of Lemma 3.1 is given by the following:

Lemma 3.2

Under assumption (24), expression (29) particularizes to:

$$A^{(3)}(\chi) = (\overline{\pi}_T, F) - (\overline{\pi}_0, F) - \int_0^T (\overline{\pi}_s, \mathcal{L} F)\, ds$$

which is exactly (25).

Proof:

Under (24):

$$(u_s, \chi^* a D v_s) = (u_s, (DF)^* a D v_s)$$

But:

$$\mathcal{L}(F v_s) = F \mathcal{L} v_s + v_s \mathcal{L} F + (DF)^* a D v_s$$

Therefore, using in particular (20):

$$
\begin{aligned}
(u_s, \chi^* a D v_s) &= (u_s, \mathcal{L}(F v_s)) - (u_s, F \mathcal{L} v_s) - (u_s, v_s \mathcal{L} F) \\
&= (v_s \mathcal{L}^* u_s - u_s \mathcal{L} v_s, F) - (u_s v_s, \mathcal{L} F) \\
&= (\dot{q}_s, F) - (q_s, \mathcal{L} F)
\end{aligned}
$$

This gives successively, using (22) again:

$$
\begin{aligned}
\int_0^T (u_s, \chi^* a D v_s)\, ds &= (q_T, F) - (q_0, F) - \int_0^T (q_s, \mathcal{L} F)\, ds \\
A^{(3)}(\chi) &= (\overline{\pi}_T, F) - (\overline{\pi}_0, F) - \int_0^T (\overline{\pi}_s, \mathcal{L} F)\, ds \qquad \square
\end{aligned}
$$

- **Study of $A^{(2)}(\eta)$**

$$
A^{(2)}(\eta) = \frac{(w_T^{(2)}, 1)}{(u_T, 1)}
$$

where:

$$
dw_t^{(2)} = \mathcal{L}^* w_t^{(2)}\, dt + h^* w_t^{(2)}\, dY_t + \eta^* u_t\, dY_t + \eta^* h u_t\, dt \qquad w_0^{(2)} = 0
$$

The "variation of constant" argument which was used for the three previous terms, does not hold here, at least in the continuous-time case. Consider instead the following partition of $[0, T]$: $0 = t_0 < t_1 < \ldots < t_N = T$, and the corresponding approximation to $(w_t^{(2)} \,;\, 0 \leq t \leq T)$:

$$
\begin{aligned}
\overline{w}_{n+1}^{(2)} &= U_{t_{n+1}}^{t_n} \overline{w}_n^{(2)} + b_n \\
b_n &\stackrel{\triangle}{=} \eta^* u_{t_n} \Delta Y_n + \eta^* h u_{t_n} \Delta t
\end{aligned}
$$

This gives successsively, using again (18):

$$
\begin{aligned}
\overline{w}_n^{(2)} &= \sum_{j=0}^{n-1} U_{t_n}^{t_{j+1}} b_j \\
(\overline{w}_n^{(2)}, \phi) &= \sum_{j=0}^{n-1} (U_{t_n}^{t_{j+1}} b_j, \phi) = \sum_{j=0}^{n-1} (b_j, V_{t_n}^{t_{j+1}} \phi) \\
&= \sum_{j=0}^{n-1} (\eta^* u_{t_j}, V_{t_n}^{t_{j+1}} \phi) \Delta Y_n + \sum_{j=0}^{n-1} (\eta^* h u_{t_j}, V_{t_n}^{t_{j+1}} \phi) \Delta t \\
(\overline{w}_N^{(2)}, 1) &= \sum_{j=0}^{N-1} (\eta^* u_{t_j}, v_{t_{j+1}}) \Delta Y_n + \sum_{j=0}^{N-1} (\eta^* h u_{t_j}, v_{t_{j+1}}) \Delta t
\end{aligned}
$$

Taking then the limit of both sides as the mesh of the partition goes to 0, gives:

$$
(w_T^{(2)}, 1) = \int_0^T (\eta^* u_s, v_s)\, dY_s + \int_0^T (\eta^* h u_s, v_s)\, ds
$$

where the stochastic integral is to be understood as a two-sided stochastic integral [6],[7]. Finally, using again (22):

$$A^{(2)}(\eta) = \frac{\int_0^T (q_s, \eta^*)\, dY_s}{(u_T, 1)} + \int_0^T (\bar{\pi}_s, \eta^* h)\, ds$$

Remarks:

- Whether or not the first term can be further simplified should be investigated, but this would definitely be out of the scope of this paper.
- As expected:

$$\mathbf{E}\left(\frac{\int_0^T (q_s, \eta^*)\, dY_s}{(u_T, 1)}\right) = \mathring{\mathbf{E}}\left(\int_0^T (q_s, \eta^*)\, dY_s\right) = 0$$

  the last equality resulting from the definition of two-sided stochastic integrals.

3.4 Conclusion

Two methods have been proposed for the computation of conditional expectations such as (11).

- Filtering gives:

$$A = \frac{(w_T, 1)}{(u_T, 1)}$$

  where $(u_t\ ;\ 0 \leq t \leq T)$ and $(w_t\ ;\ 0 \leq t \leq T)$ are solution to (14) and (27) respectively.

- Smoothing gives either:

$$(w_T, 1) = (q_0, \beta) + \int_0^T (q_s, \xi + \eta^* h)\, ds + \int_0^T (q_s, \eta^*)\, dY_s + \int_0^T \left(\chi^* a D\left[\frac{q_s}{u_s}\right], u_s\right) ds$$

$$A = (\bar{\pi}_0, \beta) + \int_0^T (\bar{\pi}_s, \xi + \eta^* h)\, ds + \frac{\int_0^T (q_s, \eta^*)\, dY_s}{(u_T, 1)} + \int_0^T \left(\chi^* a D\left[\frac{\bar{\pi}_s}{\pi_s}\right], \pi_s\right) ds$$

  where: $(q_t\ ;\ 0 \leq t \leq T)$, $(\pi_t\ ;\ 0 \leq t \leq T)$ and $(\bar{\pi}_t\ ;\ 0 \leq t \leq T)$ are given by (13), (14),(15),(16),(19).

The advantage of smoothing over filtering is that the dependence on (β, ξ, η, χ) is made explicit: provided the underlying probability does not change, evaluating A for a different set of data (β, ξ, η, χ) will not require the computation of a new infinite-dimensional object. In the filtering approach, one would have to solve another SPDE, with a different "right-hand side".

On the other hand, from the computational point of view, solving equation for the smoothing density requires the storage of the filtering density, and is therefore more expensive.

The next two sections will be devoted to the application of these two approaches to the computation of quantities related to the direct likelihood maximization, and to the EM algorithm respectively.

4 Direct maximization of the likelihood ratio

According to (4) and (12), the log-likelihood ratio $L(\theta)$ is given by any of the following expressions:

$$L(\theta) = \log(u_T^\theta, 1) = \int_0^T (\pi_s^\theta, h_\theta^*)\, dY_s - \frac{1}{2}\int_0^T (\pi_s^\theta, h_\theta^*)(\pi_s^\theta, h_\theta)\, ds$$

with (see (14)):

$$du_t^\theta = \mathcal{L}_\theta^* u_t^\theta\, dt + h_\theta^* u_t^\theta\, dY_t \qquad u_0^\theta = p_0^\theta \tag{30}$$

and:

$$\mathcal{L}_\theta \triangleq \frac{1}{2}\sum_{i,j=1}^m a^{ij}(\cdot)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^m b_\theta^i(\cdot)\frac{\partial}{\partial x_i}$$

According to (8) and (10), $\nabla L(\theta)$ belongs to the class of conditional expectations considered in Section 3, provided:

- the underlying probability is P_θ,
- the following data are used:

$$\beta_\theta = \frac{\nabla p_0^\theta}{p_0^\theta} \qquad \xi_\theta = -(\nabla h_\theta)^* h_\theta$$

$$\eta_\theta = \nabla h_\theta \qquad \chi_\theta = a^{-1}\nabla b_\theta$$

In particular:

$$\xi_\theta + \eta_\theta^* h_\theta = 0$$

The approach based on filtering gives:

$$\nabla L(\theta) = \frac{(w_T^\theta, 1)}{(u_T^\theta, 1)}$$

with $(u_t^\theta\ ;\ 0 \le t \le T)$ and $(q_t^\theta\ ;\ 0 \le t \le T)$ given respectively by (30) and (see (27)):

$$\begin{aligned} dw_t^\theta &= \mathcal{L}_\theta^* w_t^\theta\, dt + h_\theta^* w_t^\theta\, dY_t + (\nabla h_\theta)^* u_t^\theta\, dY_t + J_\theta^* u_t^\theta\, dt \\ w_0^\theta &= \nabla p_0^\theta \end{aligned} \tag{31}$$

where:

$$J_\theta \phi \triangleq J(\chi_\theta)\phi = (\nabla b_\theta)^* D\phi$$

Remarks:

- This equation is exactly what would be obtained by deriving formally equation (30), with respect to the parameter θ. This result was indeed obtained in [3], relying on the existence of a "robust" version of Zakaï equation.

- If θ is a p-dimensional parameter, then the gradient $(w_t^\theta \; ; \; 0 \leq t \leq T)$ is a p-dimensional vector: each component of this vector actually solves a SPDE which is coupled only with $(u_t^\theta \; ; \; 0 \leq t \leq T)$ and with no other component; moreover the coupling occurs only through the "right-hand side" and each of these $(p+1)$ SPDE has the same dynamics. In other words, one has to solve the same SPDE with $(p+1)$ different "right-hand side". As expected, smoothing will provide a more efficient way to deal with such a problem.

Indeed:

$$\nabla L(\theta) = (\frac{q_0^\theta}{u_0^\theta}, \nabla p_0^\theta) + \int_0^T ((\nabla h_\theta)^*, q_s^\theta)\, dY_s + \int_0^T ((\nabla b_\theta)^*\, D\left[\frac{q_s^\theta}{u_s^\theta}\right], u_s^\theta)\, ds$$

where $(u_t^\theta \; ; \; 0 \leq t \leq T)$ and $(q_t^\theta \; ; \; 0 \leq t \leq T)$ are given respectively by (30) and (see (21)):

$$\dot{q}_t^\theta + u_t^\theta \mathcal{L}_\theta \left(\frac{q_t^\theta}{u_t^\theta}\right) = \frac{q_t^\theta}{u_t^\theta} \mathcal{L}_\theta^* u_t^\theta \qquad\qquad q_T^\theta = u_T^\theta \tag{32}$$

5 The EM algorithm

According to (5) and (2), the auxiliary function $Q(\theta, \theta')$ belongs to the class of conditional expectations considered in Section 3, provided:

- the underlying probability is $P_{\theta'}$,
- the following data are used:

$$\begin{aligned}
\beta_{\theta\theta'} &= \log \frac{p_0^\theta}{p_0^{\theta'}} \\
\xi_{\theta\theta'} &= -\frac{1}{2}\left[(b_\theta - b_{\theta'})^* a^{-1}(b_\theta - b_{\theta'}) + (h_\theta^* h_\theta - h_{\theta'}^* h_{\theta'})\right] \\
\eta_{\theta\theta'} &= h_\theta - h_{\theta'} \\
\chi_{\theta\theta'} &= a^{-1}(b_\theta - b_{\theta'})
\end{aligned}$$

In particular:

$$\xi_{\theta\theta'} + \eta_{\theta\theta'}^* h_{\theta'} = -\frac{1}{2}\left[(b_\theta - b_{\theta'})^* a^{-1}(b_\theta - b_{\theta'}) + (h_\theta - h_{\theta'})^*(h_\theta - h_{\theta'})\right]$$

The approach based on filtering gives:

$$Q(\theta, \theta') = \frac{(w_T^{\theta\theta'}, 1)}{(u_T^{\theta'}, 1)}$$

with $(u_t^{\theta'} \; ; \; 0 \leq t \leq T)$ and $(w_t^{\theta\theta'} \; ; \; 0 \leq t \leq T)$ given respectively by (30) and (see (27)):

$$\begin{aligned}
dw_t^{\theta\theta'} &= \mathcal{L}_{\theta'}^* w_t^{\theta\theta'}\, dt + h_{\theta'}^* w_t^{\theta\theta'}\, dY_t + (h_\theta - h_{\theta'})^* u_t^{\theta'}\, dY_t + J_{\theta\theta'}^* u_t^{\theta'}\, dt \\
&\quad -\frac{1}{2}\left[(b_\theta - b_{\theta'})^* a^{-1}(b_\theta - b_{\theta'}) + (h_\theta - h_{\theta'})^*(h_\theta - h_{\theta'})\right] u_t^{\theta'}\, dt \\
w_0^{\theta\theta'} &= p_0^{\theta'} \log \frac{p_0^\theta}{p_0^{\theta'}}
\end{aligned}$$

where:

$$J_{\theta\theta'}\phi \triangleq J(\chi_{\theta\theta'})\phi = (b_\theta - b_{\theta'})^* D\phi$$

On the other hand, smoothing gives:

$$\begin{aligned} Q(\theta,\theta') = {} & (\overline{\pi}_0^{\theta'}, \log \frac{p_0^\theta}{p_0^{\theta'}}) + \frac{\int_0^T (q_s^{\theta'}, (h_\theta - h_{\theta'})^*)\, dY_s}{(u_T^{\theta'}, 1)} + \int_0^T ((b_\theta - b_{\theta'})^* D\left[\frac{\overline{\pi}_s^{\theta'}}{\pi_s^{\theta'}}\right], \pi_s^{\theta'})\, ds \\ & -\frac{1}{2}\int_0^T (\overline{\pi}_s^{\theta'}, \left[(b_\theta - b_{\theta'})^* a^{-1}(b_\theta - b_{\theta'}) + (h_\theta - h_{\theta'})^*(h_\theta - h_{\theta'})\right])\, ds \end{aligned} \tag{33}$$

where:

$$\pi_t^{\theta'} = \frac{u_t^{\theta'}}{(u_t^{\theta'}, 1)} \qquad \overline{\pi}_t^{\theta'} = \frac{q_t^{\theta'}}{(q_t^{\theta'}, 1)}$$

$(u_t^{\theta'}\ ;\ 0 \le t \le T)$ and $(q_t^{\theta'}\ ;\ 0 \le t \le T)$ are given respectively by (30) and (32).

Remark: It is readily seen from the last expression that the point [M] defined in the Introduction, is satisfied:

- the regularity of $Q(\cdot,\theta')$ rely in an obvious way on the existence of derivatives with respect to θ of $\log p_0^\theta$, b_θ and h_θ,
- computing the corresponding derivatives, and maximizing $Q(\cdot,\theta')$ will not involve the computation of any other infinite-dimensional object such as a conditional density.

Moreover, as was pointed out in [1], there are particular cases in which the M-step can be dealt with explicitely. This includes the case where:

- $\log p_0^\theta$ depends quadratically on θ,
- b_θ and h_θ depend linearly on θ,

since $\theta \mapsto Q(\theta,\theta')$ becomes then a quadratic form.

According to (9) and (10), $\nabla^{10}Q(\theta,\theta')$ belongs to the class of conditional expectations considered in Section 3, provided:

- the underlying probability is $P_{\theta'}$,
- the following data are used:

$$\beta_\theta = \frac{\nabla p_0^\theta}{p_0^\theta} \qquad \xi_{\theta\theta'} = -(\nabla b_\theta)^* a^{-1}(b_\theta - b_{\theta'}) - (\nabla h_\theta)^* h_\theta$$

$$\eta_\theta = \nabla h_\theta \qquad \chi_\theta = a^{-1}\nabla b_\theta$$

In particular:

$$\xi_{\theta\theta'} + \eta_\theta h_{\theta'} = -(\nabla b_\theta)^* a^{-1}(b_\theta - b_{\theta'}) - (\nabla h_\theta)^*(h_\theta - h_{\theta'})$$

The approach based on filtering gives:

$$\nabla^{10} Q(\theta, \theta') = \frac{(w_T^{\theta\theta'}, 1)}{(u_T^{\theta'}, 1)}$$

with $(u_t^{\theta'}\ ;\ 0 \le t \le T)$ and $(w_t^{\theta\theta'}\ ;\ 0 \le t \le T)$ given respectively by (30) and (see (27)):

$$\begin{aligned} dw_t^{\theta\theta'} &= \mathcal{L}_{\theta'}^* w_t^{\theta\theta'}\, dt + h_{\theta'}^* w_t^{\theta\theta'}\, dY_t + (\nabla h_\theta)^* u_t^{\theta'}\, dY_t + J_\theta^* u_t^{\theta'}\, dt \\ &\quad - \Big[(\nabla b_\theta)^* a^{-1}(b_\theta - b_{\theta'}) + (\nabla h_\theta)^*(h_\theta - h_{\theta'})\Big]\, u_t^{\theta'}\, dt \\ w_0^{\theta\theta'} &= \frac{p_0^{\theta'}}{p_0^{\theta}} \nabla p_0^{\theta} \end{aligned}$$

where:

$$J_\theta \phi \stackrel{\triangle}{=} J(\chi_\theta)\phi = (\nabla b_\theta)^* D\phi$$

Remark: Comparing with (31) one can check that:

$$\nabla^{10} Q(\theta, \theta')\,|_{\theta=\theta'} = \nabla L(\theta')$$

as expected.

As for the smoothing approach, one can use again the results of Section 3. Alternatively, one can directly differentiate with respect to θ the expression (33) for $Q(\theta, \theta')$, thus illustrating the point [**M**]. Indeed:

$$\begin{aligned} \nabla^{10} Q(\theta, \theta') &= (\overline{\pi}_0^{\theta'}, \frac{\nabla p_0^{\theta}}{p_0^{\theta}}) + \frac{\int_0^T (q_s^{\theta'}, (\nabla h_\theta)^*)\, dY_s}{(u_T^{\theta'}, 1)} + \int_0^T ((\nabla b_\theta)^* D \left[\frac{\overline{\pi}_s^{\theta'}}{\pi_s^{\theta'}}\right], \pi_s^{\theta'})\, ds \\ &\quad - \int_0^T (\overline{\pi}_s^{\theta'}, \Big[(\nabla b_\theta)^* a^{-1}(b_\theta - b_{\theta'}) + (\nabla h_\theta)^*(h_\theta - h_{\theta'})\Big])\, ds \end{aligned}$$

6 Conclusion

Two different approaches have been investigated for the MLE of partially observed diffusions. Some formulas given in [1] have been clarified, and it has been shown that smoothing is necesary to make the EM algorithm approach efficient. On the other hand, formula have been given in terms of SPDE for the computation of the original log-likelihood ratio and its gradient. (As might have been noticed, expressions related to the direct approach are given in terms of unnormalized conditional densities, whereas in the EM algorithm approach normalized conditional densities have been used).

As a consequence, it does not appear so clearly, except for some particular cases, already considered ~~in [1]~~, that the EM algorithm is faster than the direct aproach. This should be investigated on numerical exemples.

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